An Optimal Execution Model with S-shaped Temporary and Transient Market Impacts

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Abstract. When institutional investors trade a large amount of a stock in the market, the trading amount might impact the price, and the price change is called market impact (MI hereafter). In addition, their trading is always exposed to uncertain price change, which is called timing risk. They need to evaluate quantitatively MI and timing risk, and decide optimal execution strategy in consideration of the trade-off between them. Recently, the price impact models with market order are discussed under the assumption of temporary and transient MIs. Many previous studies assume temporary MI as a linear function of the amount of orders. It means that the depth of the order book is presumed to be linear with the prices in an order-driven market. Some studies, however, have shown that the depth is a convex function with the prices. In this case, temporary MI follows an S-shaped nonlinear function with the amount of orders, which is proposed by Kato (2017). In our paper, we propose the optimal execution model with S-shaped temporary and transient MIs. We conduct the sensitivity analysis in order to consider the characteristics of the optimal execution orders. In addition, we estimate the MI function and other parameters using market data and derive the optimal execution strategies for practical use. In the future task, we need to develop the method of deriving the solution stably for the nonlinear optimization problem.

Keywords: optimal execution, S-shaped temporary and transient market impacts, market order, downside risk

1. INTRODUCTION

When institutional investors trade a large amount of a stock in the market, the trading amount might impact the price, and the price change is called market impact (MI hereafter). In addition, their trading is always exposed to uncertain price change, which is called timing risk. They need to evaluate quantitatively MI and timing risk, and decide optimal execution strategy in consideration of the trade-off between them. It is common that MI is evaluated by using temporary MI and permanent (or transient) MI. The former is the temporal price changes caused by market orders, and the latter is the price change remained permanently on the fundamental prices by the orders. There are many studies which presume linear temporary and permanent MIs and derive optimal execution strategies (refer to Bertsimas and Lo (1998), Almgren and Chriss (2007), Takenobu and Hibiki (2017)). However, Bouchaud *et al.* (2006) show that the price impact is transient in the real market. In addition, Gatheral *et al.* (2011) derive the optimal execution strategy of minimizing expected cost with linear temporary and transient MIs. Alfonsi *et al.* (2012) also derive static optimal strategies in consideration of the timing risk, using variance of cost. As well, Ono *et al.* (2017) derive them using downside risk measure. As mentioned above, many previous studies use linear temporary MI. However, it is known that temporary MI is a non-linear function dependent on a limit order book. Curato *et al.* (2017) derive the optimal strategy of minimizing expected cost with concave temporary and transient MI s. Furthermore, Kato (2017) shows it is more realistic to use S-shaped temporary MI in order driven markets like the Japanese stock market. He derives the optimal execution strategy in a restricted situation by solving HJB equation in consideration of only S-shaped temporary MI. But Kato (2017) has shortcomings that the timing risk and transient MI are not introduced in the model.

In our paper, we discuss the optimal execution model with S-shaped temporary and transient MIs and downside risk, and we formulate the optimal execution problem in a discrete time.

Our contributions are the following two points. First, we propose the stochastic optimization model involving S-shaped temporary and transient MIs under the framework of Ono *et al.* (2017). We find the S-shaped temporary MI gives an incentive to execute them earlier or later dependent on the target amount of orders. In addition, we compare the proposed model with previous models and find the features of our model which is more useful in the discrete time setting of the large number of periods. Second, we estimate the MI function and other parameters using market data and derive the optimal execution strategies of a real stock. In addition, we compare it with the other execution strategies calculated using the previous studies, and we find the practical usefulness of our model. We show the differences with some of previous studies as follow.

Table 1: Comparison with some of previous studies

	Takenobu and Hibiki (2017)	Alfonsi *et al.* (2012)	Ono *et al.* (2017)	Kato (2017)	Our study (2018)
Temporary MI	Linear	Linear	Linear	S-shaped	S-shaped
Transient MI	Constant	Decay function	Decay function	×	Decay function
Risk measure	Downside risk (LPM)	Variance	Downside risk (LPM)	×	Downside risk (LPM)
Estimation using market data	×	×	○	×	○

2. OPTIMAL EXECUTION PROBLEM

We set up the problems with reference to Almgren and Chriss (2007), Alfonsi *et al.* (2012), and Ono *et al* (2017). We presume we hold a block of shares X of a single security. And the initial price is P_0. We need to sell a stock in the market by time horizon T. We divide T into K intervals of length $\tau(=T/K)$. We plan to hold x_k shares at time $k\ (k=1,\ldots,K)$, and therefore we shall sell $x_{k-1}-x_k$ between $k-1$ and k. The average rate of trading during period k is $v_k = x_{k-1}-x_k/\tau$.

2.1 S-shaped Temporary Market Impact

In order driven markets, using linear temporary MI likewise many previous studies is the same as to assume bloke-type limit order books. Kato (2017) shows that evaluating MI as S-shaped MI is more realistic. This means that the shapes of limit order books are inverted V-shaped (See Figure 1). Generally, this S-shaped temporary MI function is formulated as Eq. (1).

$$h(x) = \begin{cases} h_0 \dfrac{\pi_2}{\pi_1} \bar{x}_0^{\pi_2-\pi_1} x^{\pi_1} & (0 < x \le \bar{x}_0) \\ h_0 \left(\dfrac{\pi_2}{\pi_1} - 1\right) \bar{x}_0^{\pi_2} + h_0 x^{\pi_2} & (\bar{x}_0 \le x) \end{cases} \quad (1)$$

Where, $h(x)$ is the value of temporary MI when the amount of orders is x. π_1 and π_2 are curvatures of concave and convex functions, respectively. $\bar{x}_0$ represents the amount of orders at the inflection point of the function. The linear, convex and concave temporary MIs can be expressed as special cases of the S-shaped temporary MI.

We estimate the temporary MI functions for all brands of TOPIX100 listed Tokyo Stock Exchange using the tick data in 2016. We examine whether the shapes of functions of large stocks are S-shaped.

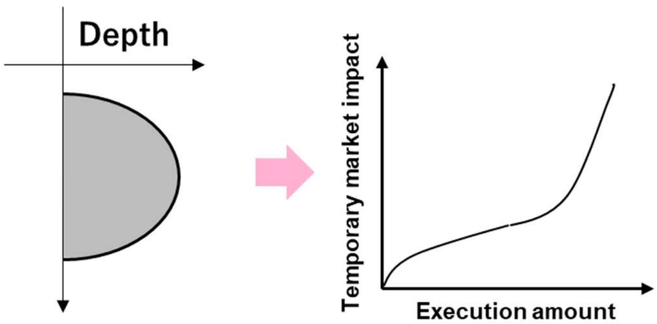

Figure 1: Limit order book and temporary MI

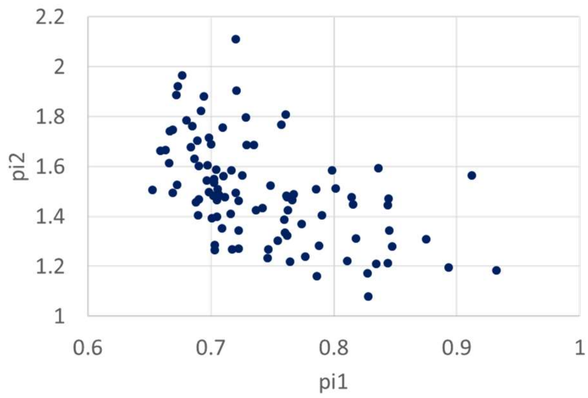

Figure 2: Curvatures of S-shaped transient MIs

We plot the estimated parameters (π_1, π_2) for TOPIX100 stocks in Figure 2. This indicates that the shapes of limit order books of these stocks are inverted V-shaped and temporary MI functions are S-shaped because we can estimate all $\pi_1 \le 1$ and $\pi_2 \ge 1$. The median is $(\pi_1, \pi_2) = (0.72, 1.48)$ and FANUC Corporation (security code: 6954), has one of the most extreme S-shaped temporary MIs $(\pi_1, \pi_2) = (0.72, 2.11)$.

2.2 S-SHAPED TEMPORARY AND TRANSIENT MARKET IMPACT

In this research we use transient market impact which is suitable for the real market, proposed and showed by Bouchaud *et al.* (2006). We define the transient MI with reference to Alfonsi *et al.* (2012). When τv shares are executed, the S-shaped temporary MI $h(\tau v)$ (See Eq. (1)) occurs and decay to the value multiplied by $G : [0, \infty) \longrightarrow [0,1]$, which is called decay kernel. We assume G as a power function as the Eq. (2). In this case, we can formulate temporary and permanent MIs by using a constant G, in many previous studies, such as Almgren and Chriss (2007) and Takenobu and Hibiki (2017).

$$G_u = (1 + \lambda u \tau)^{-\rho_p} \qquad (\rho_p, \lambda \geq 0) \tag{2}$$

Therefore, the MI at time k derived from the execution at time $u-1$ $(k \geq u)$ can be formulated as,

$$MI\big((k-u+1)\tau\big) = h(\tau v_u) G_{k-u+1}. \tag{3}$$

2.3 PRICE DYNAMICS

We presume that price process follows the arithmetic Brownian motion. Therefore, the evolution of the fundamental price P_k and execution price $\tilde{P}_k$ involving MI can be formulated as follows.

$$P_k = P_0 + \sigma\sqrt{\tau}\sum_{u=1}^{k}\xi_u - \tau\sum_{u=1}^{k} h(\tau v_u) G_{k-u+1} \tag{4}$$

$$\tilde{P}_k = P_{k-1} - h(\tau v_k) \tag{5}$$

We represent the random price change as $\sigma\sqrt{\tau}\xi_u$ using daily standard deviation, σ, and uncertain fluctuations in period $[u-1, u]$, $\xi_u \sim N(0,1)$.

2.4 EXECUTION COST

We evaluate the total cost of trading, or implementation shortfall, for selling the amount of security which is the difference between the initial market value and the final capture of the trade derived using trading policy. It is expressed as

$$C_K = XP_0 - \sum_{k=1}^{K}(x_{k-1} - x_k)\tilde{P}_k \tag{6}$$

$$= \sum_{k=1}^{K}\sum_{u=1}^{k} G_{k-u} h(x_{u-1} - x_u)(x_{k-1} - x_k)$$

$$-\sigma\sqrt{\tau}\sum_{k=1}^{K-1}\xi_k x_k \tag{7}$$

where, $x_0 = X, x_K = 0$. The first term of Eq. (7) shows the MI cost, and the second term indicates the timing risk.

2.4 DOWNSIDE RISK

We evaluate the timing risk by the first order lower partial moment (LPM, hereafter). LPM is an expected value of total cost(C_K) beyond a target cost (C_G). The LPM can be formulated using expected cost ($\bar{C}_K$) and variance of cost (σ_C) as,

$$LPM(C_K) = \int_{C_G}^{\infty}(C_K - C_G) f(C_K) dC_K$$

$$= \{\phi(Q) + Q\Phi(Q)\}\sigma_C, \tag{8}$$

$$Q = (\bar{C}_K - C_G)/\sigma_C, \tag{9}$$

$$\bar{C}_K = \sum_{k=1}^{K}\sum_{u=1}^{k} G_{k-u} h(x_{u-1} - x_u)(x_{k-1} - x_k), \tag{10}$$

$$\sigma_C^2 = \sigma^2 \times \frac{1}{K}\sum_{k=1}^{K-1} x_k^2, \tag{11}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ represent a density function and cumulative distribution function of standard normal distribution, respectively.

2.4 OPTIMAL EXECUTION MODEL WITH S-SHAPED TEMPORARY AND TRANSIENT MARKET IMPACTS

We formulate an optimal execution model with S-shaped temporary and transient MIs (called S-shaped and transient model or proposed model) of minimizing the sum of the expected total cost and LPM multiplied by the risk aversion as follows.

(1) Notations

a) Parameters

K: number of periods $(k = 1, \dots, K)$

γ: risk aversion coefficient

C_G: target cost

X: total amount of stock to be executed

h_0: S-shaped MI coefficient

x_0: amount of orders at the inflection point

π_1, π_2: curvatures of concave and convex

ρ_p, λ: Decay speeds of transient MI

b) Variables

x_k: residual fraction of order held at time k, determined at time $k-1$ $(k = 1, \dots, K)$

$LPM(C_K)$: first-order lower partial moment of total cost

(2) Formulation

$$\min \quad \bar{C}_K + \gamma LPM(\bar{C}_K) \tag{12}$$

subject to

$$\bar{C}_K = \sum_{u=1}^{k} G_{k-u} h(x_{u-1} - x_u)(x_{k-1} - x_k) \ (k = 1, \dots, K)$$

$$(x_0 = X, x_1^{(j)} = x_1, x_K^{(j)} = 0) \tag{13}$$

$$LPM(C_K) = \{\phi(Q) + Q\Phi(Q)\}\sigma_C \tag{14}$$

$$Q = (\bar{C}_K - C_G)/\sigma_C \tag{15}$$

$$\sigma_C^2 = \sigma^2 \times \frac{1}{K}\sum_{k=1}^{K-1} x_k^2 \quad (16)$$

$$x_k \le x_{k-1}\ (k = 1, \dots, K) \quad (17)$$

$$h(x) = \begin{cases} h_0 \dfrac{\pi_2}{\pi_1} \bar{x}_0^{\pi_2-\pi_1} x^{\pi_1} & (0 < x \le \bar{x}_0) \\ h_0 \left(\dfrac{\pi_2}{\pi_1} - 1\right) \bar{x}_0^{\pi_2} + h_0 x^{\pi_2} & (\bar{x}_0 \le x) \end{cases} \quad (18)$$

$$G_{k-u} = (1 + \lambda(k-u)\tau)^{-\rho p} \quad (19)$$

Eq. (13) is the expected total cost. Eqs. (14) to (16) are used for the calculation of LPM. Eq. (17) is the constraint of non-increasing in time for residual orders. Eq. (18) is the S-shaped MI function. Eq. (19) is the function of the decay kernel of transient MI. Moreover, when we use a fixed value instead of $h(x)$. This model is the same as the execution model with linear temporary and transient MI proposed by Ono *et al.* (2017). When we replace G_{k-u} with a constant value ($G_0 = 1$, otherwise $\bar{G}$), it is the same as the execution model with linear temporary and permanent MI proposed by Takenobu and Hibiki (2017).

4. NUMERICAL ANALYSIS

We derive optimal execution strategy with proposed model using hypothetical data in order to clarify the characteristics of the optimal execution orders. All of the problems are solved using Numerical Optimizer (Ver 18.1) — mathematical programming software package developed by NTT DATA Mathematical System, Inc. on Windows 10 personal computer which has Corei7-6700K, 4.00GHz CPU and 32GB memory. We set parameters of basic case as, $K = 5, T = 1, X = 5{,}000, C_G = 36{,}000, \sigma = 50, \gamma = 0$. We adopt risk neutral execution strategy to illustrate the characteristics of S-shaped temporary MI model easily.

4.1 SETTING TEMPORARY MARKET IMPACT

We generate three kinds of hypothetical order books which have 250 orders on average in each price, with reference to the result of Section 2.1. We estimate linear, normal S-shaped, extreme S-shaped temporary MI functions based on the three kinds of order books, respectively in Table 2.

Table 2: Parameters for temporary MI

	Linear	Normal S-shaped	Extreme S-shaped
$\bar{x}_0$	1000	1000	1000
h_0	0.004	5.6E-05	2.4E-09
π_1	1	0.84	0.53
π_2	1	1.53	2.82

4.2 USENG A CONSTANT DECAY KERNEL

When we use linear temporary and transient MI model with a constant decay kernel $\bar{G}$ and solve the problem, the optimal execution strategy is trading in equal size. Therefore, we conduct numerical analysis with transient MI model with constant decay kernel ($G_{k-u} = \bar{G} = 0.5$) to clarify the features of S-shaped temporary MI model. Where $\bar{G}$ represents the residual rate of temporary MI. We show the optimal strategy in Figure 3 by using abovementioned base parameters. It shows that the solutions with S-shaped temporary MI are different from those of linear MI. Furthermore, the larger the curvature of S-shaped MI is, the larger the difference becomes.

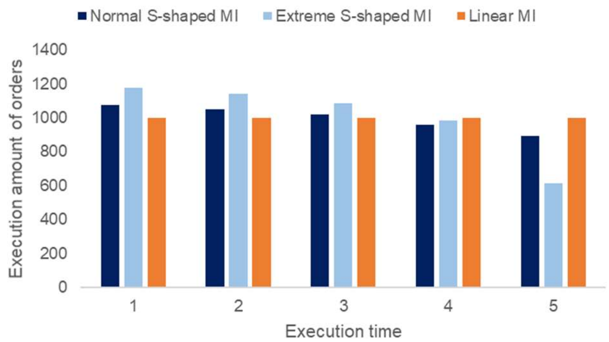

Figure 3: Optimal executions with base parameters

We conduct the sensitivity analysis to discuss the features of S-shaped temporary MI in this section. We derive optimal execution strategies with normal S-shaped MI using some sorts of X and $\bar{G}$ and show the fractions of orders in each time in Table 2.

Table 2: Optimal execution strategies with normal S-shaped temporary MI

Execution time	$\bar{G}$=0.25			$\bar{G}$=0.75		
	$X = 100$	$X = 5{,}000$	$X = 90{,}000$	$X = 100$	$X = 5{,}000$	$X = 90{,}000$
1	21.3%	20.8%	18.7%	27.3%	22.3%	16.6%
2	20.6%	20.5%	19.3%	22.7%	21.6%	17.9%
3	20.0%	20.1%	19.9%	19.2%	20.7%	19.5%
4	19.3%	19.6%	20.6%	16.5%	18.9%	21.6%
5	18.8%	19.0%	21.4%	14.3%	16.5%	24.5%

We find the characteristics of the optimal execution strategy with S-shaped temporary MI in Table 2 as below.

(1) X and execution strategy

When X/K is larger than $\bar{x}_0$ (1,000 shares in this case), the early amounts of orders are getting smaller.

(2) $\bar{G}$ and execution strategy

When we use large $\bar{G}$, which shows the influence of the investor in the market, the difference from the strategy of trading in equal size is getting huge.

From the above (1) and (2), we find that estimated S-shaped MI has following features. We execute the large amount of orders in early periods with concave temporary MI, whereas we execute the large amount of orders in later periods with convex temporary MI. In addition, the larger constant decay kernel, the larger the difference of executions among the periods. These are unique features of S-shaped temporary MI model we have never observed in the linear temporary MI.

4.3 TRANSIENT MARKET IMPACT

Similarly, we derive optimal execution strategies with S-shaped temporary and transient MIs using base parameters in Figure 4, where $\rho_p = 3, \lambda = 1$ are set as additional base parameters. We have an incentive of executing large amount of orders in the earlier and later periods due to getting the benefit of the transient MI, shown by Alfonsi *et al.* (2012), and Figure 4 shows the feature.

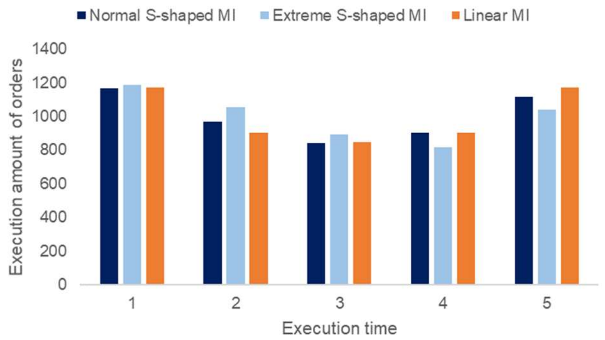

Figure 4: Optimal executions with transient MI using based parameters

4.3.1 RISK AVERSE STRATEGY

We derive optimal execution strategy in consideration of downside risk with various risk aversions in Figure 5. We find the features as well as previous studies that the amount of earlier orders increases and the risk (LPM) can be reduced. In addition, we derive optimal execution strategies with various target costs C_G in Figure 6. When C_G is extremely small or large, the effect of LPM affecting the objective function is small because the LPM is close to the expected total cost minus target cost or zero. Therefore, the strategy becomes almost close to that of minimizing only expected total cost.

Figure 5: Optimal executions with various γ

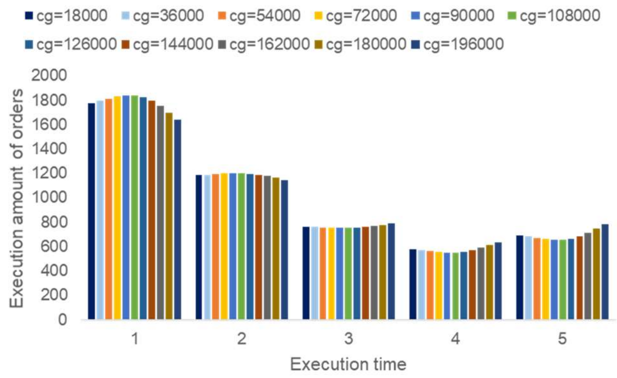

Figure 6: Optimal executions with various C_G

4.3.2 SENSITIVITY ANALYSIS OF K

We set four kinds of $K = 5, 10, 25, 50$ and derive risk neutral optimal execution strategies with normal S-shaped temporary and transient MIs and show the results including actual number of orders, expected total costs, and optimal strategies on Table 3 and Figure 7. Moreover, we also show the results with optimal strategies with linear model and the strategy of trading in equal size. In our paper, "actual number of executions" is defined as the number of times with actually executed nonzero orders.

The strategies in Figure 7 show the features of S-shaped temporary MI we already mention in Section 4.3, but when we increase K, the actual number of orders executions does not increase and the objective function value also does not improve (rather getting worse, see Table 3).

This shows that the strategy of executing the orders collectively in early and later periods is better than executing at all time points over periods, due to the concave function. Furthermore, we find that this problem is ill-posed through various analyses, and therefore it is difficult to derive a unique solution numerically for a large number of K. Developing the solution method is our further research. Table 3 shows the expected costs of optimal solution of the proposed model are better than those of linear model and the trading strategy in equal size. In addition, the expected costs of linear model and the trading strategy in equal size do not improve as the number of periods increases. This is the noteworthy feature of S-shaped impact different from linear impact[1]. This is likely to show that the discrete time setting is suitable for the continuous time setting for constructing the model involving the S-shaped impact.

Table 3: Results with various K

K	5	10	25	50
Actual number of executions	5	9	15	15
Proposed model	35,332	32,217	31,144	31,226
Linear model	35,371	32,407	32,660	34,279
Equal size	35,689	33,279	34,539	37,022

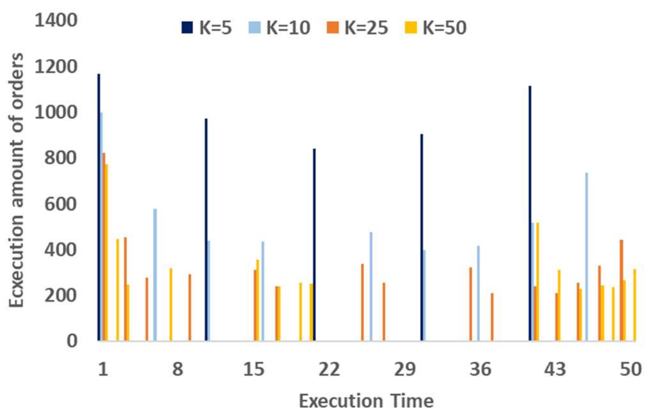

Figure 7: Optimal executions with various K

5. ANALYSIS USING MARKET DATA

Table 4: Results of comparison to previous studies (unit: basis point)

	γ = 0		γ = 1	
K	vs Linear	vs Equal	vs Linear	vs Equal
5	0.4	4	87	856
10	57	69	42	1,899
50	1,800	1,900	200	4,400

We estimate the MI function and other parameters using market data and derive the optimal execution strategies for practical use. Softbank (9984), which is a largescale stock listed with the first section of Tokyo Stock Exchange, is supposed to be executed and the estimated parameters are shown as follows, $(h_0, \bar{x}_0, \pi_1, \pi_2) = (1.9E-05, 12080, 0.72, 1.27)$, $(\rho_p, \lambda) = (0.21, 500)$, $\sigma = 107.45$ (yen). We solve the problem with $K = (5, 10, 50), X = 128{,}000$ (1billion yen), $C_G = 5{,}024{,}000$ (about 0.5% of total execution price) and show the results of the comparison to the strategies with previous model in Table 4. This shows the practical usefulness of our model.

[1] It is known well that expected costs of linear model and the trading strategy in equal size get smaller as the number of periods K increases.

6. CONCLUSION

We propose the optimal execution model with S-shaped temporary and transient MIs and show the characteristics and usefulness of the model through various analysis using hypothetical data and real market data.

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