Asset Allocation with Factor-based Risk Parity Strategy

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Abstract. Asset allocation strategy is important to manage assets effectively. In recent years, risk parity strategy has become attractive to academics and practitioners in place of traditional mean-variance approach. Risk parity strategy determines the individual allocation for asset classes in order to equalize their contributions to overall portfolio risk, it implies risk parity strategy is sensitive to the choice of asset's universe and ignores common risks among assets. In this regard, Roncalli and Weisang (2016) proposed the use of "risk factors" instead of asset classes. This approach aims at "true diversification" by allocating portfolio risk to each factor. In this paper, we construct the factor-based risk parity portfolio with macroeconomic factors such as stock, interest rate and inflation following Roncalli and Weisang (2016) and Qian (2012). We use global bonds (government, aggregate corporate and high-yield), stocks (developed and emerging country) and REIT data. We find this approach can reduce portfolio risk which cannot be considered in the conventional asset-based risk parity portfolio. However, we also find this method has some shortcomings. The first point is the difficulty of obtaining a unique solution from an optimization problem. The second point is stability, the relationship between assets and factors is changing dynamically in time series, optimal portfolio weights are also changing dynamically. Therefore, we construct the portfolio in consideration of not only factor diversification but also asset diversification. We propose a new risk parity method which balances between asset and factor diversification. We implement eighteen years backtest, we find our method decreases risks which represent standard deviation and downside risk, and it has less turnover and increases stability. We show our new method reduces risk and has practical advantages.

Keywords: Finance, Asset Allocation, Risk Parity Portfolio, Factor Investing, Optimization

1. INTRODUCTION

The asset allocation strategy is extremely important to manage assets effectively. The standard asset allocation model is the mean-variance model by Markovitz. However, this model is extremely sensitive to the change in parameters. From 2000, many researchers and practitioners have shown interests in the risk parity portfolio instead of the traditional mean-variance model. The portfolio equalizes each asset's contribution to overall portfolio risk. It seeks portfolio is not affected by specific asset variance and aims at the stable asset management.

While explicitly pursuing diversification, these methodologies may lead to solutions with hidden risk concentration. The optimal asset-based risk parity portfolio depends on which assets are included in the portfolio.

In this regard, Roncalli and Weisang (2016) proposed the use of "risk factors" instead of asset classes for the risk parity portfolio. It is based on the idea that asset class returns and variability are driven by common risk factors, and aims at "true diversification" by allocating portfolio risk to each factor. However, we find this method has some shortcomings. The first point is the difficulty of obtaining a unique solution from an optimization problem. The second point is stability, the relationship between assets and factors is changing dynamically in time series, optimal portfolio weights are also changing dynamically.

Therefore, we construct the portfolio in consideration of not only factor diversification but also asset diversification. We propose a new risk parity method which balances between asset and factor diversification. Our methodology can explicitly control both factor and asset diversification. It expects the optimal portfolio weights are stable and have many practical advantages. We consider investing in the global financial indexes and clarify these characteristics.

Our paper is organized as follows. In Section 2, we explain the basic risk parity portfolio methodology. In Section 3, we explain the factor-based risk parity portfolio and its shortcomings. We propose a new risk parity method. In Section 4, we derive the portfolio allocation using real financial data

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and discuss its characteristics. In Section 5, we implement eighteen years backtest and evaluate the performance of our new method. We find our method has many practical advantages.

2. RISK PARITY PORTFOLIO

We define each asset's risk contribution. The most commonly used definition is based on Euler's homogeneous function theorem. It is defined as follows,

$$RC(\mathcal{A}_i) = x_i \frac{\partial R(\boldsymbol{x})}{\partial x_i} \tag{1}$$

where $R(x)$ is the portfolio risk, x_i is the portfolio weight to asset i, and $\mathcal{A}_i$ is the set to asset i.

When we use the standard deviation as a risk measure $\left(R(\boldsymbol{x}) = \sigma_P = \sqrt{\boldsymbol{x}^\top \Sigma \boldsymbol{x}}\right)$, Equation (1) can be replaced as,

$$RC(\mathcal{A}_i) = x_i \frac{(\Sigma \boldsymbol{x})_i}{\sqrt{\boldsymbol{x}^\top \Sigma \boldsymbol{x}}} \tag{2}$$

where Σ is the covariance matrix of asset returns.

Then, we satisfy the following equation.

$$R(\boldsymbol{x}) = \sum_{i=1}^{N} RC(\mathcal{A}_i) \tag{3}$$

Equation (3) shows that the total portfolio risk equals the sum of each asset risk contribution by Euler's homogeneous function theorem. The risk parity strategy equalizes its risk contribution across all assets and the portfolio risk can be equally diversified to each asset.

Roncalli (2013) said this risk parity portfolio's advantages are

1. It defines a portfolio that is well diversified in terms of risk and weights.
2. It does not necessary any expected returns estimation.
3. It is less sensitive to small changes in the covariance matrix than the mean-variance portfolio.

3. FACTOR-BASED RISK PARITY PORTFOLIO

The asset-based risk parity portfolio equalized its risk contributions across all assets. It depends on the portfolio asset's universe and leads to hidden risk concentration. Many asset classes load on the same risk factor (e.g. equity risk), it generates an asset-based risk parity portfolio with very concentrated risk factor exposure. Due to this problem, Roncalli and Weisang (2016) show the risk parity portfolio with risk factors instead of assets. It means that each asset class can have embedded loadings on several factors and can share some of them with one or more other asset classes that are apparently distinct.

3.1. Formulation

We assume the following linear factor model,

$$\boldsymbol{R} = \boldsymbol{A}\boldsymbol{F} + \boldsymbol{\varepsilon} \tag{4}$$

where $\boldsymbol{R}$ is the vector of asset returns, $\boldsymbol{F}$ is the vector of factor returns and $\boldsymbol{A}$ is the 'loadings' matrix.

We satisfy,

$$\boldsymbol{y} = \boldsymbol{A}^\top \boldsymbol{x} \tag{5}$$

where $\boldsymbol{y}$ is the vector of portfolio's risk factors exposures.

Following these relationships, the risk contribution to risk factor j is,

$$RC(\mathcal{F}_j) = y_j \frac{\partial R(\boldsymbol{x})}{\partial y_j} = (\boldsymbol{A}^\top \boldsymbol{x})_j \cdot \left(\boldsymbol{A}^+ \frac{\partial R(x)}{\partial \boldsymbol{x}^\top}\right)_j \tag{6}$$

where $\mathcal{F}_j$ is the set to factor j and A^+ is the Moore-Penrose inverse of A. When we use the standard deviation as a risk measure, Equation (6) can be replaced as,

$$RC(\mathcal{F}_j) = (A^\top \boldsymbol{x})_j \cdot \left(A^+ \frac{\Sigma \boldsymbol{x}}{\sqrt{\boldsymbol{x}^\top \Sigma \boldsymbol{x}}}\right)_j \tag{7}$$

Then, we solve an optimization problem that the contributions to each factor are the same for all factors. The first constraint expresses the sum of portfolio weights is 1. The second constraint expresses long-only portfolio constructed.

$$\begin{aligned} &\textbf{Minimize} && \sum_{j=1}^{M} \left(\frac{RC(\mathcal{F}_j)}{\sigma_P} - \frac{1}{M}\right)^2 \\ &\textbf{subject to} && \sum_{i=1}^{N} x_i = 1 \\ &&& 0 \le x_i \le 1 \end{aligned} \tag{8}$$

However, this optimization problem is not convex around the minimum optimal value, therefore it is difficult to obtain a unique solution. Figure 1 shows the objective value of Problem (8). We can find there are feasible regions of portfolio weights where risk contributions are allocated equally to all factors.

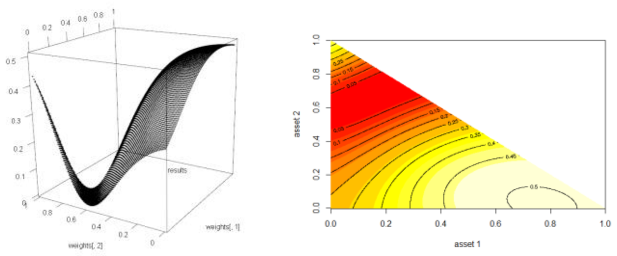

Figure 1: Objective function value

3.2. New risk parity method

In this paper, we propose a new risk parity method which balances between asset and factor diversification as follows.

$$\begin{aligned} \textbf{Minimize} \quad & \lambda \sum_{j=1}^{M} \left(\frac{RC(\mathcal{F}_j)}{\sigma_P} - \frac{1}{M} \right)^2 \\ & + (1-\lambda) \sum_{i=1}^{N} \left(\frac{RC(\mathcal{A}_i)}{\sigma_P} - \frac{1}{N} \right)^2 \quad (9) \\ \textbf{subject to} \quad & \sum_{i=1}^{N} x_i = 1 \\ & 0 \le x_i \le 1 \end{aligned}$$

We construct the portfolio in consideration of not only factor diversification but also asset diversification. Diversification in asset includes the difference in market participants and differences by country. In Equation (9), $\lambda = 0$ shows the asset-based risk parity portfolio and $\lambda = 1$ shows the factor-based risk parity portfolio. In our model, we can control both asset and factor diversification by adjusting λ.

3.3. Portfolio Diversification Index (PDI)

Rudin and Morgan (2006) proposed an index which measures the number of unique investments in a portfolio. It is useful to measure diversification benefits across the universe. PDI is defined as follows,

$$PDI = 2 \cdot \sum_{i=1}^{N} i \cdot RS_i - 1, \qquad RS_i = \frac{\Lambda_i}{\sum_{j=1}^{N} \Lambda_j} \qquad (10)$$

where Λ_i is the eigenvalue associated with the i-th principal component and RS_i is the relative strength of the i-th principal component. When $PDI = 1$, it indicates diversification is effectively impossible. On the other hand, when $\mathrm{PDI} = N$, it indicates all assets are perfectly uncorrelated. PDI shows diversification potential of a set of the universe.

We use the PDI as a weight between asset and factor diversification. We minimize the risk concentration of assets and factors which are weighted by the potential of diversification of each universe.

$$\begin{aligned} \textbf{Minimize} \quad & \frac{PDI(\mathcal{F})}{M} \sum_{j=1}^{M} \left(\frac{RC(\mathcal{F}_j)}{\sigma_P} - \frac{1}{M} \right)^2 \\ & + \frac{PDI(\mathcal{A})}{N} \sum_{i=1}^{N} \left(\frac{RC(\mathcal{A}_i)}{\sigma_P} - \frac{1}{N} \right)^2 \quad (11) \\ \textbf{subject to} \quad & \sum_{i=1}^{N} x_i = 1, \qquad 0 \le x_i \le 1 \end{aligned}$$

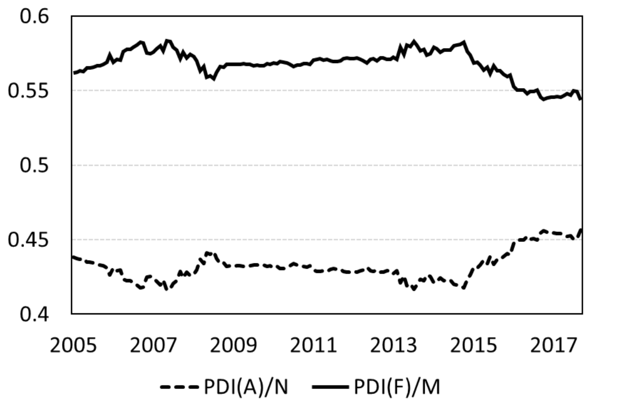

Figure 2: Asset and factor PDIs

Figure 2 shows the standardized PDI estimated in 60 months. The factor PDI is higher than the asset PDI because the correlations among factor returns are lower than those of asset returns. When the financial crisis occurred in 2008, the correlations of factor returns have increased and the factor PDI has declined.

4. BASIC ANALYSIS

We conduct the basic analysis. We use the following index returns of seven asset classes and three factors.

Asset return:

(1) Developed country stock: MSCI World
(2) Emerging country stock: MSCI Emerging Market
(3) Government bond: Bloomberg Global Treasury
(4) Corporate bond: Bloomberg Global Aggregate Corporate
(5) High-Yield: Bloomberg Global High-Yield
(6) Commodity: Bloomberg commodity
(7) REIT: S&P Global REIT Index

Period: October 2000 – April 2018
Currency: USD

Factor return:

Qian (2013) said that *Risk Parity portfolio should have a balanced risk contribution from three sources: (1) equity risk; (2) interest rate risk; (3) inflation risk*. In reference to Qian(2013), we use these three factors to construct the factor-based risk parity portfolio.

(1) Equity: MSCI ACWI
(2) Interest rate: Bloomberg Barclays Global-Aggregate minus inflation risk factor return
(3) Inflation risk: Bloomberg Barclays global Inflation-Linked minus Bloomberg Barclays Global Treasury

Table 1. Result of Basic Analysis

Portfolio weight $(= x_i)$						
λ	**0**	**0.25**	**0.5**	**0.75**	**1**	**PDI**
Developed Stock	9.88%	8.53%	7.22%	6.13%	5.05%	6.96%
Emerging Stock	6.56%	5.71%	5.06%	4.56%	2.11%	4.94%
Government Bond	28.54%	36.35%	39.25%	40.58%	42.98%	39.63%
Cooperate Bond	21.84%	23.76%	25.53%	26.67%	24.22%	25.83%
High-Yield Bond	14.20%	11.99%	11.43%	11.29%	4.74%	11.38%
Commodity	10.76%	5.67%	3.54%	2.82%	4.39%	3.31%
REIT	8.21%	8.00%	7.96%	7.95%	16.51%	7.96%
Percent risk contribution to asset $(= RC(\mathcal{A}_i)/\sigma_P)$						
λ	**0**	**0.25**	**0.5**	**0.75**	**1**	**PDI**
Developed Stock	14.29%	12.63%	10.63%	8.91%	6.74%	10.21%
Emerging Stock	14.29%	12.75%	11.26%	10.06%	4.12%	10.96%
Government Bond	14.29%	22.09%	26.10%	28.37%	28.71%	26.71%
Cooperate Bond	14.29%	17.56%	19.97%	21.52%	18.20%	20.38%
High-Yield Bond	14.29%	12.65%	12.25%	12.16%	4.58%	12.21%
Commodity	14.29%	7.36%	4.51%	3.57%	5.18%	4.20%
REIT	14.29%	14.96%	15.28%	15.42%	32.47%	15.32%
Percent risk contribution to factor $(= RC(\mathcal{F}_i)/\sigma_P)$						
λ	**0**	**0.25**	**0.5**	**0.75**	**1**	**PDI**
Equity risk	51.06%	44.70%	39.95%	36.53%	33.33%	39.09%
Interest rate risk	3.29%	17.94%	26.50%	31.07%	33.33%	27.76%
Inflation risk	47.06%	39.03%	35.26%	34.11%	33.33%	34.86%

4.1. Result

Using these data, we get the following loadings matrix A.

$$A = \begin{pmatrix} 0.98 & -0.04 & -0.07 \\ 1.19 & 0.38 & 0.66 \\ -0.02 & 1.18 & 1.01 \\ 0.07 & 0.95 & 1.27 \\ 0.37 & 0.50 & 1.11 \\ 0.33 & 0.76 & 1.53 \\ 0.71 & 0.84 & 1.30 \end{pmatrix}$$

Table 1 shows optimal portfolio weight, risk contribution to asset and risk contribution to factor from the factor-based risk parity portfolio to each λ.

At $\lambda = 0$, which is the asset-based risk parity portfolio, the risk contributions to assets are perfectly equal, but risk contributions to factors show the risk is concentrated. The proportions of equity risk and inflation risk are large to the total risk. On the other hand, at $\lambda = 1$, which is the factor-based risk parity portfolio, the risk contributions to factors are perfectly equal, but the portfolio allocations of government bond, cooperate bond and REIT are large. Moreover, we find an optimal solution is unstably derived at $\lambda = 1$ because it is dependent on the initial value.

The loadings matrix A shows that commodity has a very large exposure to inflation risk. As λ increases, the portfolio weight to commodity decreases to lower the risk contribution of inflation risk. In contrast, government bond has relatively high exposure to interest rate risk. As λ increases, the portfolio weight to government bond increases to raise the risk contribution to interest rate risk.

The asset PDI is 4.50 and the factor PDI is 2.42. It corresponds to $\lambda = 0.65 (= 4.50/(4.50 + 2.42))$in Equation (9). The risk contribution can be well-balanced between assets and factors in the PDI strategy.

We find that the asset with excessive exposure to risk factor tends to be underweighted. The PDI strategy gives well-balanced portfolio weight in consideration of risk contribution between assets and factors.

5. BACKTEST

It is well-known that the risk of financial asset return and their correlations are time-varying. It causes the different optimal factor-based risk parity portfolio over time. We implement the backtest under the following setting. Suppose we invest seven assets, as well as the basic analysis in Section 4. The portfolio is rebalanced on the first day of each month, and the loadings matrix and risk contributions are estimated using a rolling window of sixty months.

Table 2 reports out-of-sample average returns and standard

Table 2. Result of backtest

λ	0	0.25	0.5	0.75	1	PDI
Annual average return	3.72%	3.85%	3.97%	3.89%	3.41%	3.96%
Annual standard deviation	8.96%	8.41%	8.09%	7.91%	8.11%	8.03%
95%-VaR	3.81%	3.58%	3.42%	3.22%	3.27%	3.36%
95%-CVaR	6.17%	5.73%	5.49%	5.39%	5.61%	5.46%
Sharpe ratio	0.3917	0.4340	0.4654	0.4666	0.3950	0.4679
CVaR ratio	0.5687	0.6369	0.6856	0.6839	0.5713	0.6887
Maximum drawdown	-30.52%	-27.96%	-26.26%	-25.66%	-27.72%	-26.03%
Turnover	1.52%	2.48%	3.27%	4.31%	9.21%	3.52%

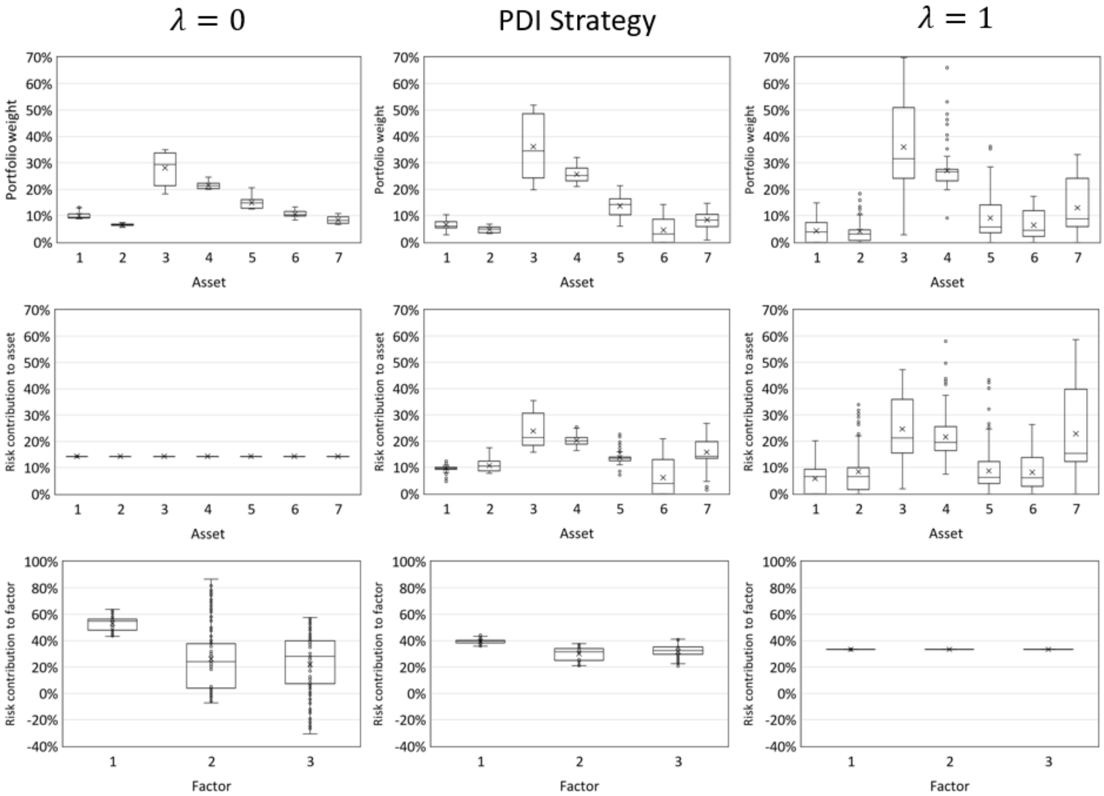

Figure 3: Portfolio weight and risk contributions to assets and factors

deviations on the portfolios. As λ increases, the standard deviation of the portfolio decreases. We evaluate investment efficiency between return and risk using Sharpe ratio, or

$$SR = \frac{\overline{r_P} - r_f}{\sigma_P} \tag{12}$$

where $\overline{r_P}$ is a portfolio return and r_f is a risk free rate. We use one month JPY LIBOR as risk free rate.

We find the PDI strategy has the highest Sharpe ratio. This result implies the PDI strategy gives portfolio weights so that it can capture the potential of asset and factor diversification appropriately.

Table 2 also reports down-side risks. First, VaR (Value at Risk) represents the potential maximum loss on a given confidence level α ($\alpha = 0.95$ in this paper).

$$VaR(\alpha) = \min_{1-\alpha}\{V: P[-r_P > V] \leq (1-\alpha)\} \tag{13}$$

where r_P is a portfolio return. Second, CVaR (Conditional VaR) is defined as the average loss beyond the VaR.

$$CVaR(\alpha) = E[-r_P \mid -r_P \geq VaR(\alpha)] \tag{14}$$

Then, CVaR ratio defined as

$$CVaR\ ratio(\alpha) = \frac{\overline{r_P} - r_f}{CVaR(\alpha)} \tag{15}$$

Finally, we employ maximum drawdown (MDD) which is the maximum loss from a peak to a trough of a portfolio, before a new peak is attained.

$$MDD = \frac{\min_{\tau \in (0,T)} \left(\min_{t \in (0,\tau)} P(\tau) - P(t) \right)}{\max_{t \in (0,T)} P(t)} - 1 \qquad (16)$$

where T is the total number of testing periods and $P(t)$ is the portfolio value at time t.
Table 2 shows that not only standard deviation but also downside risks are decreasing as λ increases. It implies that the tail risk of the portfolio is also reduced.

At last, we examine the stability of portfolio weights. In many cases, portfolio rebalancing requires a fee. It is one of the important factors when we evaluate practical performance. We define portfolio turnover as the average sum of the absolute value of the trades across the *N* assets.

$$\text{Turnover} = \frac{1}{T-1} \sum_{t=1}^{T-1} \sum_{i=1}^{N} |x_i^{t+1} - x_i^t| \qquad (17)$$

where x_i^t is a portfolio weight to asset i at time t. Table 2 shows that increasing λ leads to a high turnover. The factor-based risk parity portfolio($\lambda = 1$) has the largest turnover. Figure 3 shows the portfolio weight and risk contributions of each portfolio. On the left side is the asset-based risk parity portfolio ($\lambda = 0$). The risk contributions to assets are always constant, but those to factors are sometimes concentrated. On the right side is the factor-based risk parity portfolio ($\lambda = 1$). We find the portfolio weight dynamically changes. On the middle is the PDI strategy. It gives the well-balanced portfolio involving the diversification between assets and factors. We conclude that increasing the weight to factor diversification gives the reduction of portfolio risk involving tail risk, but the increase in portfolio turnover. The PDI strategy which can balance the factor and asset diversification provides better performance and practical advantage with lower turnover.

5. CONCLUSION

In this paper, we discuss the factor-based risk parity portfolio which combines factor investing and risk parity strategy. Both of them have attracted attention in recent years. This strategy aims at stable asset management that is not affected by the market environment. We propose a new risk parity strategy which balances both asset and factor diversification. We develop the method in order to improve some shortcomings in the previous research.

In the basic analysis, we find that the asset with higher exposure to risk factor tends to decrease its portfolio weight. Our method gives the well-balanced portfolio in consideration of risk contribution to assets and factors. We also implement backtest under the investment on global financial assets. We find the investment strategy involving the factor-based approach leads to reduce portfolio risk and improve the efficiency of asset management.

In the future research, we need to compare with different risk parity strategies using downside risk measure.

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