A Study on Optimal Limit Order Strategy using Multi-Period Stochastic Programming considering Nonexecution Risk

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Abstract. Our paper discusses optimal trading strategy of stock using limit orders for institutional investors who would like to minimize execution cost. The limit order could satisfy their needs due to the smaller market impact than market order. However, the limit order has risk of not being filled which is called nonexecution risk. Therefore, we must consider this risk for implementing the limit order strategy as well as market impact and timing risk. Some previous literatures assume the execution probability distribution is independent on the order size, ignoring their relationship. According to some empirical analyses, executing a larger amount of limit order is more difficult than a smaller amount. This relationship is required to be considered in execution strategy. They also force nonexecuted limit order to be replaced as market order only at maturity. The reorder strategy proposed in our paper allows investors to replace nonexecuted limit orders as new limit orders. This strategy is determined considering the trade-off among nonexecution risk, market impact, and timing risk. The nonexecution risk can be considered more appropriately than other models through this strategy. The characteristics in our paper are as follows.
1. We derive the optimal limit order strategy to solve the multi-period stochastic programming problem.
2. We allow the replacement of nonexecuted order.
3. We estimate market impact and execution probability distribution from tick by tick data in Tokyo Stock Exchange. These estimates are used for solving the optimization problem.
4. We evaluate the nonexecution risk for the large amount of limit order empirically, and show the optimal strategy of reducing execution cost with the nonexecution risk.
We examine the characteristics and usefulness of the model through the sensitivity analyses with respect to various parameters. The results show that the large order is placed from the first period under some parameter settings whereas splitting the target order into smaller pieces becomes optimal in other cases.

Keywords: Multi-period Stochastic Programming, Optimal Execution Strategy, Limit Orders

1. INTRODUCTION

Institutional investors have tendency to hold position for longer period than day trader or HFT (High Frequency Trader). To do this, their position is required to be checked and rebalanced regularly for market change. When they rebalance their portfolio, there is huge risk of impact over stock exchange market. Traders in security companies who get such requests from institutional investors need to make execution strategy in order to reduce cost. When the huge order enters to the market, market price could change market price negatively. Consequently, execution cost becomes large. This risk is called market impact. There are many previous literatures discussing optimal split of huge orders to avoid this risk. Generally, market impact is able to be mitigated through splitting the huge target amounts into smaller pieces. However, if it takes long time to liquidate huge amounts, unpredictable price change could cause large cost. This is called timing risk.

The limit order setting both price and size has possibility to execute at lower cost than using market orders. However, limit order has risk of not being filled, called nonexecution risk. In order to make execution strategy using limit order, institutional investors have to consider trade-off relationship among market impact risk, timing risk, and nonexecution risk.

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Recently, Agliardi and Gencay [1] discussed optimal execution strategy with respect to limit orders. They assume the probability of the limit order being filled, called fill rate, is dependent on the order size, and the market impact of limit order does not exist. They derive their optimal strategy assuming this relationship under the exponential distribution. This relationship is mentioned in some literatures such as Kumaresan-Krejic [2010], [2015]. They usually use trading data of hedge fund that could track their order submission to execution. However, there are no previous studies using tick by tick trading data concerning whether the assumption is reasonable. In addition, some previous literatures discuss the execution strategy for large institutional investors based on the existence of market impact of limit order. And also, their strategy does not allow us to trade intraday limit orders for unexecuted limit orders, but it only allows us to execute market orders at maturity for those unexecuted limit order. We could change this trading rule so that unexecuted limit order can be replaced as limit order from the next period. This reorder rule allows institutional investors to reduce unexecuted limit order volume aggressively.

In this research, we propose the estimation models for the market impact and the fill rate. These parameters are estimated empirically using tick by tick data of Tokyo Stock Exchange. Finally, the estimated parameters are implemented into optimization. The optimal execution strategy is expressed as the replacement of non-executed limit order.

Our paper is organized as follows. We introduce an execution strategy model and its concept in Section 2. The market impacts of limit order are estimated in Section 3. We estimate the fill rate in Section 4. Using these estimated parameters, we derive the optimal execution strategy and conduct the sensitivity analyses in Section 5. Section 6 concludes our paper.

2. Optimal Execution Strategy Model

We construct the model involving the market impact, timing risk, and nonexecution risk in order to derive the optimal limit order strategy. The strategy is calculated in the multi-period stochastic programming approach using Monte Carlo simulation method, proposed by Hibiki [4]. This model gives a single optimal decision in each period under the simulation paths generated over the multiple periods.

Only the main constraints are expressed due to space limitation.

$$\min_{x_1,\cdots,x_N} \quad E[C_{N+1}] + \gamma \cdot CVaR[C_{N+1}] \tag{1}$$

$$s.t. \quad w_k^{(i)} = \tau_k^{(i)} z_k^{(i)} \tag{2}$$

$$\tau_k^{(i)} \sim TND(mu_k^{(i)}\left(z_k^{(i)}\right), s_k^{(i)}\left(z_k^{(i)}\right)) \tag{3}$$

$$z_k^{(i)} = \min(y_{k-1}^{(i)}, x_k) \tag{4}$$

$$z_{N+1}^{(i)} = w_{N+1}^{(i)} = y_N^{(i)} \tag{5}$$

$$y_k^{(i)} = y_{k-1}^{(i)} - w_{k-1}^{(i)} \ \ (k = 1, \cdots, N+1) \tag{6}$$

$$\ln P_k^{(i)} = \ln P_{k-1}^{(i)} + LMI \cdot z_k^{(i)} + \sigma \xi_k^{(i)} \tag{7}$$

$$\ln P_{N+1}^{(i)} = \ln P_N^{(i)} + MMI \cdot z_{N+1}^{(i)} + \sigma \xi_k^{(i)} \tag{8}$$

$$C_k^{(i)} = w_k^{(i)}\left(\ln P_k^{(i)} - \ln P_0^{(i)}\right) + C_{k-1}^{(i)} \tag{9}$$

$$CVaR[C_{N+1}] = a_\beta + \frac{1}{(1-\beta)I}\sum_{i=1}^{I} u^{(i)} \tag{10}$$

$$a_\beta - C_{N+1}^{(i)} + u^{(i)} \le 0 \tag{11}$$

$$0 \le u^{(i)} \tag{12}$$

$$\sum_{k=1}^{N+1} w_k^{(i)} = y \tag{13}$$

$$y_0^{(i)} = y, y_{N+1}^{(i)} = 0, P_0^{(i)} = P_0, C_0^{(i)} = 0 \tag{14}$$

I is the total number of simulation paths whose index is i, and N is the total number of trading period whose index is k. Therefore, we generate I simulation paths and derive the optimal strategy over N periods.

Unpredictable price change is denoted by $\xi_k^{(i)}$ and reflect timing risk, where $\xi_k^{(i)}$ follows independent and identical standard normal distribution. We assume that the price change is based on geometric Brownian motion, and expressed by lognormal distribution. The initial asset price is denoted by P_0. Market impact of limit and market orders are assumed to be linear to the order size, and these parameters are denoted by LMI and MMI, respectively. Linear price impacts imply the flat-shaped order book. Market order is only executed at maturity in order to meet the target amount completely.

The target volume is denoted by y. The posted order volume $z_k^{(i)}$ is given by the minimum value of the remaining order volume at (k-1)-th period $y_{k-1}^{(i)}$ and upper bound of order size x_k which is a decision variable of the model. If the remaining order volume $y_k^{(i)}$ is relatively large and posted, it could cause huge market impact. This often may occur during the early periods. In this case, posting only the upper bound could be better. On the other hand, when remaining order volume $y_k^{(i)}$ is sufficiently small and posted, it might not make large impact. This is a typical situation in the later periods, and posting the remaining amount could be optimal in considering timing risk and nonexecution risk.

The execution percentage $\tau_k^{(i)}$ are determined under the limit order $z_k^{(i)}$. This fill rate is dependent on the order volume $z_k^{(i)}$ and follows truncated normal distribution (TND) ranged between zero and one. The parameters of TND, or mean $m_k^{(i)}$ and standard deviation $s_k^{(i)}$, are estimated through the regression analysis discussed in the later section. Using the estimated mean $m_k^{(i)}$, standard deviation $s_k^{(i)}$, and a random number generated from uniform distribution $u_k^{(i)}$, a fill rate $\tau_k^{(i)}$ can be calculated. If $\tau_k^{(i)}$ is below one, unexecuted order volume exists and the re-order must be executed forward. Reducing the remaining order volume allows the small market order to be executed in the strategy.

The execution cost is defined based on the difference between a market price in each period and an initial price. The objective function is sum of expected cumulative execution cost and CVaR of the cost multiplied by the risk aversion coefficient γ.

Constraint (13) is imposed on the target volume, and the last constraints are set for the boundary conditions.

We need to estimate market impact coefficients LMI, MMI, and TND parameters mu and sd in the execution strategy model. We discuss it from the next section.

3. Market Impact
3.1 Estimation Model

In the previous literature by Cont et al. [3], the order flow imbalance, which is sum of limit, cancel, and market orders, is used to explain price change. We use the following notations in the model: Lb for limit buy, Cb for cancel buy, Mb for market buy, Ls for limit sell, Cs for cancel sell, and Ms for market sell order. The price change is expressed as,

$$\Delta P_t = \beta_0 + \beta_1 \cdot OFI_t \tag{15}$$

$$OFI_t = Lb_t - Cb_t - Ms_t - Ls_t + Cs_t + Mb_t \tag{16}$$

The expression in Eq. (16) is formulated due to the effect for price change. For example, the inflow of market sell order may consume best bid price so that price goes down. Therefore, the sign of Ms can be negative.

The model has two assumptions as follows. At first, it assumes that there are no correlations among six order types. However, the correlations among six order types might exist. According to our empirical data analysis using Tokyo Stock Exchange tick by tick data, the correlations are around 0.4 between cancel order and other limit and market orders, respectively. This shows that unexecuted limit order is canceled and replaced as a new market order or a limit order which could be executed more easily. Therefore, these six order types are correlated each other, and we also find the assumption is not reasonable. The second assumption is that all six order types equally affect the price change. However, it is not reasonable, because the market impact of market order is, for example, larger than limit order in general.

We improve the model in order to analyze the data under the more reasonable assumptions. At first, we delete the cancel order from the regression in order to remove large correlations among order types. Second, we divided the OFI into order types so that we can have different impacts for limit and market orders. The regression model is constructed as follows.

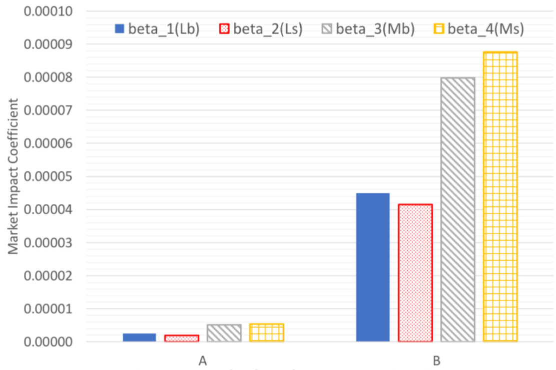

Figure 1: Result of Market Impact Estimation

$$\Delta P_t = \beta_0 + \beta_1 Lb_t - \beta_2 Ms_t - \beta_3 Ls_t + \beta_4 Mb_t \tag{17}$$

The logarithm of mid-price change ΔP_t is expressed with these four orders. The signs of $\beta_i (i = 1, \dots, 4)$ must be positive.

3.2 Data

Tick by tick data of Tokyo Stock Exchange in 2016 is used for estimation. We rank the Nikkei 225 stocks in order of trading value. Then, the assets are divided into four groups. We select two stocks for estimation. Softbank (9984) is chosen as the most liquid stock, and Sumitomo Dainippon Pharma (4506) is chosen as the least liquid stock. We call these stocks “Stock A” and “Stock B”, respectively.

3.3 Result of Estimation

Figure 1 shows the market impacts estimated in the proposed regression model. The coefficients of stock A is on the left side, and stock B on the right side. They are adjusted in an order unit of 1,000,000-yen to normalize the difference of price range.

Adjusted R^2 of stock A is 0.498 and stock B is 0.480. These are relatively large and the regression model is suitable to describe price change.

Other four order types are significant and the regression model itself is significant as well. However, the intercepts in the model are not statistically significant. According to Figure 1, the market impact of the least liquid stock is far larger than the most liquid stock. Furthermore, the market impact of market order is larger than that of limit order. We test hypothesis that the four order types have the same impact, and reject it. Therefore, the assumption of previous literature is not correct. The result implies that the limit order strategy could reduce execution cost more than the market order strategy. Finally, the market sell order has the largest impact among four order types. We could say that investors are afraid of large negative event to stock price, and react aggressively.

We use β_1 as LMI and β_4 as MMI for the optimization model shown in Section 2.

4. Fill Rate
4.1 Estimation Model

We refer to the Cont and Kukanov [2] model for estimating the fill rate. The fill rate is estimated by tracking incoming limit order L_t, outflowing cancel order C_t and market order M_t, and initial depth V_{t-1}. Their estimation model is expressed as in Eq. (18).

$$p(L_t) = \min(\max(\frac{M_t + C_t - V_{t-1}}{L_t})) \tag{18}$$

However, the cancel order is not modeled properly in their research. Their model assumes that cancel order is limited to currently placed limit order. This assumption is true in the short execution time horizon, such as a few seconds. However, we need to examine the assumption when we focus on execution strategy in the long time horizon, such as 10 to 15 minutes.

The estimation model introduced here considers cancel order more precisely. We track incoming limit order L_t, cancel order C_t, and market order M_t until the best price is renewed to outside of the spread. When the best price is moved to outward, incoming limit orders L_t are assumed to be cancelled or liquidated by market order M_t. The cancel order C_t is assumed to be included in both depth V_{t-1} and incoming limit order L_t. We define relative percentage of C_t included in L_t as a relative size of L_t to the sum of L_t and V_{t-1}. This ratio is denoted by LR as in Eq. (19). The amount of executed limit order is obtained using depth V_{t-1} and amount of depth cancelled $(1 - LR) \cdot C_t$. In reference to Cont and Kukanov [2], the value of LR is assumed to be zero. The fill rate is expressed as in Eq. (20).

$$LR = \frac{L_t}{V_{t-1} + L_t} \tag{19}$$

$$p(L_t) = \min\left(\max\left(\frac{M_t + (1 - LR) \cdot C_t - V_{t-1}}{L_t}, 0\right), 1\right) \tag{20}$$

4.2 Result of Estimation

The fill rates of limit buy order are estimated with order size. They are adjusted in an order unit of 100,000-yen to normalize the difference of price range. Figure 2 shows the graph of the estimation result where the mean of the estimated fill rate is on vertical axis and the order size is on horizontal axis.

According to Figure 2, we could conclude that the fill rate declines as the order size increases and the assumption of Agliardi-Gencay [1] is reasonable. And also, the fill rate

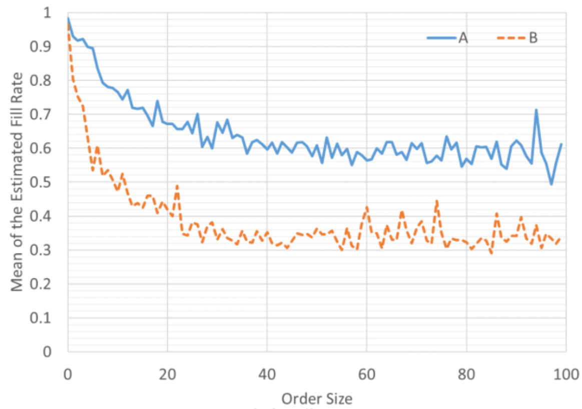

Figure 2: Result fo Fill Rate Estimation

of stock B is more sensitive and decreases when order size increases. Since stock B is less liquid, the impact of the same order size to the daily transacted volume is relatively higher than stock A. Therefore, the incoming market order is smaller and large limit order of stock B is hard to be liquidated.

We need to consider the fact that the increase in order size leads to the decline in the fill rate for managing nonexecution risk, because the relationship affects optimal execution strategy. To model this relationship, we regress the estimated fill rate using exponential function. We use $p(L_t) = a \cdot \exp(b \cdot L_t)$ model. Table 1 shows the result of this regression analysis.

Table 1: Coefficients of Fill Rate Regression

	a	b	$Adj.R^2$
A	0.813	-0.003	0.260
B	0.770	-0.016	0.079

We also find the standard deviation of fill rate is also exponentially declined to the increase in order size, and therefore we model a standard deviation of fill rate as an exponential function. We make TND random number using the standard deviation together with the mean.

5. Numerical Analysis of Optimal Execution Strategy
5.1 Basic Analysis

We set the basic parameters: risk aversion coefficient $\gamma = 0.5$, number of trading period $N = 6$, and number of simulation path $I = 10{,}000$. Target order volume y is set to 100. An initial price P_0 is given by the average daily end price in 2016.

Figure 3 illustrates the result for stocks A and B, optimized under the basic parameters. The horizontal axis shows trading period from period 1 (t1) to period 6 (t6) and terminal market order (MO). The vertical axis shows average posted order unit which is the average of $z_k^{(i)}$ in each period.

According to Figure 3, all target volume is placed from the first period in stock A. In this case, all unexecuted order unit has been replaced since the second period. The optimal solutions do not become large orders in the later periods and market order by reducing unexecuted order volume aggressively. On the other hand, around a half of

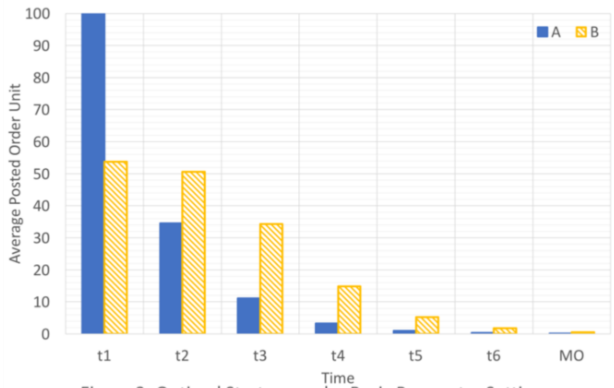

Figure 3: Optimal Strategy under Basic Parameter Setting

the target order volume is placed in the first period in stock B. The optimal solutions are subject to the upper bound of order size x_k in this case. There is no period when huge order is placed, because of splitting the target into smaller pieces.

The difference in the optimal strategies can be explained using the risks introduced previously. In case of stock A, market impact is relatively small. And also, nonexecution risk of limit order is smaller than stock B. Therefore, timing risk becomes critical. By placing huge order from the first period, the optimal strategy enables the investor not to be influenced by unpredictable market price change. On the other hand, we need to focus on nonexecution risk of huge limit order and market impact for stock B. If the market impact is large, there is incentive to split the order into smaller pieces so that the impact does not occur. What is more, Figure 2 shows that the large limit order is harder to be executed than smaller limit orders in case of stock B. Splitting the target is the optimal strategy in stock B in consideration of these two risks.

5.2 Risk Aversion Coefficient

Next, we analyze the sensitivity of optimal strategy to the risk aversion coefficient γ. As we have seen in the previous basic case analysis, stock A has a strong incentive to control timing risk due to small market impact. Therefore, we focus on stock B in order to highlight the sensitivity to the parameters. Here, we set three kinds of risk aversion coefficients which are 0, 1, and 10. The optimal average posted order units are shown in Figure 4.

The large order is placed in setting small γ and small order placement in large γ. When γ is large, risk measure CVaR is largely weighted. When a large amount is ordered in the market, the range of unexecuted order volume becomes wider than smaller order and execution cost range becomes wider as well. Consequently, CVaR

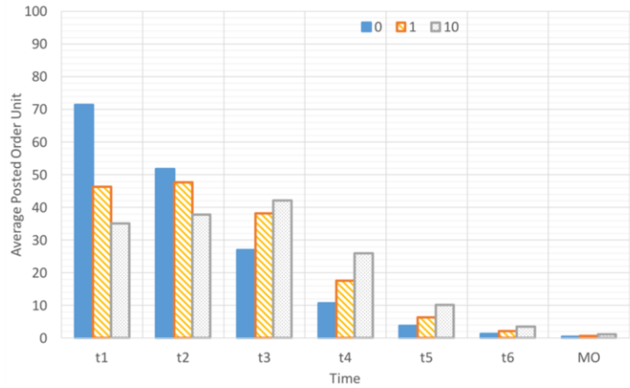

Figure 4: Optimal Strategy under Risk Aversion Coefficient settings

becomes larger. Therefore, splitting the order amounts is the optimal strategy in larger γ to prevent the cost from fluctuating.

5.3 Market Impact of Limit Order

We examine the sensitivity of market impact of limit order (LMI). We focus on stock B as in the previous section. We set ten kinds of LMIs, which are the basic parameter of LMI (=0.000045) multiplied by zero and ten. Figure 5 shows optimal strategies.

The initial order size is large when LMI is small and it becomes small when LMI becomes large. In case of smaller

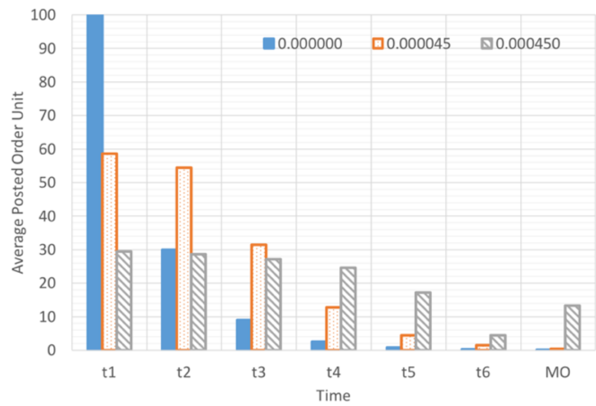

Figure 5: Optimal Strategy under LMI settings

LMI, the investor does not have to pay attention to market impact risk. Therefore, timing risk becomes critical and all target amounts are placed in the first period. On the other hand, if LMI becomes large, there is no incentive to use limit order and splitting target is optimal.

5.5 Nonexecution Risk and Execution Cost

Finally, we examine the execution cost for two kinds of optimal strategies with/without nonexecution risk, which are called Strategy A and B, respectively. Strategy A considers the relationship between order size and fill rate which is estimated in the previous section, whereas Strategy B assumes that the fill rate is independent to order size. We get the same results for Strategy A and B in case of stock A since all remaining order volume is placed in each period. Therefore, we focus on stock B, and show optimal strategies in Figure 6.

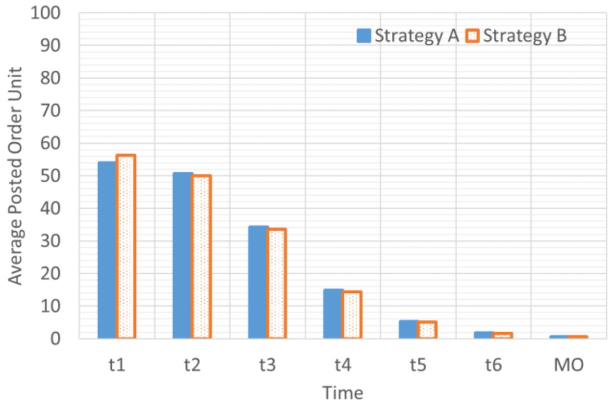

Figure 6: Strategy with Nonexecution Risk (Stock B)

The initial order size is slightly smaller in case of Strategy A. Since executing a large amount of limit order is more difficult than a small amount, the optimal strategy of splitting into pieces is chosen to avoid a large order. The objective function (Obj), expected execution cost (E[Cost]), and CVaR of execution cost is summarized in Table 2.

Table 2: Nonexecution Risk and Execution Cost

	Obj	E[Cost]	CVaR
Strategy A	1.269	0.789	0.960
Strategy B	1.272	0.789	0.965

The slight decrease in CVaR contributes to the decrease in the objective function value by managing nonexecution risk of large limit order. We conclude that the proper management could reduce execution cost.

6. Conclusion

Optimal execution strategy using limit order is proposed in this research. Market impact and fill rate are estimated through empirical data analysis. According to the estimation, market impact of liquid stock is smaller than illiquid stock and fill rate of liquid stock is higher than illiquid stock. These parameters highlight the difference of optimal strategy. All target volume is placed in liquid stock due to timing risk, whereas the target is divided into pieces in less liquid stock considering market impact and nonexecution risk of large limit order.

In this research, the fill rate of limit order is estimated using tick by tick data. The limit order strategy for institutional investors is optimized with the estimated parameters. According to the estimation, we find that the limit order is hard to be executed if its order size becomes large especially in case of less liquid stocks. We could also estimate this relationship through regression analysis using exponential function. We derive the optimal execution strategy in consideration of the relationship. In case of high liquid stock, all target volume is placed due to timing risk and nonexecution risk. However, the optimal strategy is to split the target into smaller pieces in case of less liquid stock because of the market impact and another nonexecution risk which relates to dependency of fill rate to order size.

Market order could also be used with limit order to execute more efficiently. And also, market impact of limit order should carefully be discussed but these are future extensions of this research.

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