Estimation decay kernel of transient market impact using multi-dimensional Hawkes process

Tomoki Okada
Graduate School of Science and Technology
Keio University, Yokohama, Japan
Tel: (+81) 80-6568-2358, Email: t-okada0811@keio.jp

Norio Hibiki
Faculty of Science and Technology
Keio University, Yokohama, Japan
Tel: (+81) 45-566-1635, Email: hibiki@ae.keio.ac.jp

Abstract. When institutional investors trade a large amount of stock, they expose the market impact risk that stock price moves largely in undesired direction due to the imbalance of supply and demand in the market. They trade the total order amount little by little in order to reduce market impact. On the other hand, the timing risk is also exposed owing to uncertainty of stock price. Considering the trade-off between the market impact risk and timing risk, they need to plan the optimal execution strategy. Ono, Hibiki and Sakurai(2017) develop an optimal execution strategy based on a transient impact model in which market impact gradually decays over time. Ono et al.(2017) employ the method of Bouchaud et al.(2004) so as to estimate a decay kernel of transient impact from the market data. However, this method has the difficulty of estimating it well, and therefore a more robust method is required. In our study, we use the price model based on a Hawkes process proposed by Bacry et al.(2013) and analyze the method of identifying a decay kernel of transient impact based on Amaral and Papanicolaou(2017). In the empirical analysis, we calibrate the parameters of Hawkes process with exponential kernel from the market data and demonstrate that the liquidity or intraday seasonality does not affect resilience of transient impact significantly because it only affects the baseline intensity of Hawkes process. Finally, we show that the method of using Hawkes process is more robust, compared with the method of Bouchaud et al.(2004).

Keywords: Execution strategy, Market impact, Hawkes process

1. INTRODUCTION

The institutional investors such as insurance companies and pension funds have to trade a large amount of stock when they construct and rebalance their portfolio. Then, the stock price moves largely in undesired direction due to the imbalance of supply and demand in the market. Such price moving is called market impact, which is managed as one of risks in the execution of large orders. They can reduce market impact by trading the total order amount little by little. They are also exposed to another risk that they cannot execute orders at the desirable price due to the uncertainty of the stock price. It is called timing risk, and there is a trade-off relationship between market impact risk and timing risk. Considering this relationship, it is important to plan trading strategies of minimizing total execution costs.

There are a lot of previous studies concerning the optimal execution strategy using market impact model. Most studies such as Almgren and Chriss(2001), Takenobu and Hibiki(2016) derive the execution strategy based on temporary/permanent impact model. Recently, it is said that the transient impact model is a more realistic model than the temporary permanent impact model. Ono et al.(2017) solve an optimal execution problem under the existence of transient impact such that temporary impact is gradually recovered over time.

When we develop the transient impact model to derive the optimal execution strategy in practice, it is necessary to estimate the decay kernel of transient impact appropriately from the market data. Ono et al.(2017) estimate the transient impact nonparametrically based on the method of Bouchaud et al.(2004), and model the decay kernel parametrically using exponential and power functions. However, the method of Bouchaud et al. (2004) has a disadvantage for the low liquidity stock because of inducing a strong bias.

The transient impact is defined as the decay of stock price after the market impact, and we need the statistically effective number of execution samples of the large amount of orders by a specific trader in order to obtain the estimates. However, most actual data is anonymously available. One method of estimating from anonymous data is developed under a Hawkes process. The Hawkes process has been studied long for modeling earthquakes. Recently, a model using a Hawkes process has drawn attention for financial high frequency data analysis. Bacry et al. (2013) propose a high frequency price model using a Hawkes process in order to solve the problem caused by the microstructure noise where the volatility and covariance cannot be estimated well. It has been widely applied in other fields of finance. Amaral and Papanicolaou (2017) studied decay of market impact under the condition that an external impact is involved in the mutually excited two-dimensional symmetric Hawkes process with exponential kernel. In this paper, we analyze the decay of market impact using the Hawkes process in reference to Amaral and Papanicolaou (2017),.

2. MODEL

First, we explain the theory of Hawkes process. Next, we define a mid-price process as the difference of counting process between the events of rising price and falling price under mutual excited symmetric two-dimensional Hawkes process. Finally, we explain the theory of identifying transient impact based on the behavior after giving an external impact to the price process.

2.1 Theoretical background of Hawkes process

2.1.1 Definition of Hawkes process

First, we introduce the concept of "intensity". This is equivalent to a hazard rate, which represents the probability of occurrence of instantaneous events in credit risk and survival time analysis.

Definition 1. Intensity

Assuming that a counting process $N(t)$ is a point process adapting to filtration $\mathcal{F}_t$, the left-continuous intensity is defined as in Eq. (1).

$$\lambda(t|\mathcal{F}_t) = \lim_{h \downarrow 0} \mathbb{E}\left[\frac{N(t+h)-N(t)}{h}|\mathcal{F}_t\right] \quad (1)$$

As can be seen from Eq.(1), the intensity function represents probability density such that the next event is observed under the information given up to time t. Let us assume that the filtration $\mathcal{F}_t$ always exists for $N(t)$ and denote $\lambda(t|\mathcal{F}_t)$ as $\lambda(t)$ to avoid complications. A Poisson process is one of the simplest point processes.

Definition 2. Poisson process

Let λ be real number more than or equal to zero, and a point process defined by Eqs.(2) and (3) is called a Poisson process.

$$P[N(t+h)-N(t)=1|\mathcal{F}_t] = \lambda h + o(h) \quad (2)$$

$$P[N(t+h)-N(t)>1|\mathcal{F}_t] = o(h) \quad (3)$$

The Poisson process has the properties that each time point is stochastically independent of all of other time points and the duration between successive events follows exponential distribution with the parameter λ.

Hawkes (1971) propose a self-exciting point process, which is more generalized than a Poisson process, and called a Hawkes process.

The intensity λ of the Poisson process is a constant value or a time dependent and deterministic function, whereas the intensity of the Hawkes process is excited by past events. We explain how the intensity is defined in the Hawkes process.

Definition 3. Intensity of multi-dimensional Hawkes process

We define *D*-dimensional Hawkes process $\{N^i(t)\}_{1\le i\le D}$ and intensity corresponding to $\{\{\lambda^i(t)\}_{1\le i\le D}$, which has probabilistic dynamics in Eq. (4),

$$\lambda^i(t) = \mu^i + \sum_{j=1}^{D}\int_0^t \varphi^{ij}(t-s)dN^j(s), \quad \forall i \in [1,D], \quad (4)$$

where μ^i is a baseline intensity of the sequence, $\varphi^{ij}(t-s)$ is an excitation function representing the degree of influence of the sequence i receiving from sequence j and is called a Hawkes kernel. Representative Hawkes kernels are as follows.

$$\text{Exponential kernel}: \varphi^{ij}(t-s) = \alpha^{ij}e^{\beta^{ij}(t-s)} \quad (5)$$

$$\text{Power}-\text{law kernel}: \varphi^{ij}(t-s) = \alpha^{ij}(\gamma^{ij}+t-s)^{-\beta^{ij}} \quad (6)$$

2.1.2 Calibration

The maximum likelihood method is a most general calibration method for estimating a Hawkes process. When a Hawkes kernel is exponential, a log-likelihood function $\ln\mathcal{L}$ of a multi-dimensional Hawkes process is showed by Ogata (1978) as,

$$\ln\mathcal{L}(N(t)_{t\le T}) = \sum_{i=1}^{D}\ln\mathcal{L}^i(\{N^i(t)\}_{t\le T})$$

$$= \sum_{i=1}^{D}\left\{-\mu^i T + \sum_{j=1}^{D}\sum_{t_k^j}\frac{\alpha^{ij}}{\beta^{ij}}\left(e^{-\beta^{ij}\left(T-t_k^j\right)}-1\right)\right.$$

$$\left. + \sum_{t_l^i}\ln\left[\mu^i\sum_{j=1}^{D}\alpha^{ij}R^{ij}(l)\right]\right\}, \quad (7)$$

where $R^{ij}(l)$ follows a recursive formula as in Eq. (8)

$$R^{ij}(l) = \sum_{t_k^j<t_l^i} e^{-\beta^{ij}\left(t_l^i-t_k^j\right)} \quad (8)$$

$$= \begin{cases} e^{-\beta^{ij}(t_l^i-t_{l-1}^i)}R^{ij}(l-1) + \sum_{t_{l-1}^i\le t_k^j<t_l^i} e^{-\beta^{ij}\left(t_l^i-t_k^j\right)} & \text{for } i\ne j \\ e^{-\beta^{ij}(t_l^i-t_{l-1}^i)}(1+R^{ij}(l-1)) & \text{for } i=j \end{cases}$$

As the number of dimensions increases, the number of parameters increases in the squared order. Furthermore, a log-likelihood function of Hawkes process is non-convex, and therefore it is difficult to estimate parameters in general. A Nelder-Mead method (1965) has been a well-known algorithm of obtaining global optimal solutions effectively without gradient information only using the function values. Therefore, the method has been often used for the calibration of Hawkes process. On the other hand, Lu and Abergel (2017) point out that the differential evolution method proposed by Storn and Price (1996) is more effective. It adopts an algorithm of searching the optimal solution using metaheuristics. Our paper also uses the differential evolution method as the optimization algorithm when calibrating the parameters of Hawkes process from the market data in the empirical analysis.

2.2 Hawkes-based price model

Next, we explain the price model using a Hawkes process, proposed by Bacry et al. (2013). They derive the price process considering the microstructure by modelling up and down events of mid-price with two-dimensional Hawkes process. The model may be able to cope with the microstructure noise where the volatility and covariance cannot be properly estimated due to remarkably short intervals in high frequency data analysis. This price process is collectively referred to as a Hawkes-based price model in this paper. Especially, we employ the symmetric Hawkes-based pricing model which takes into account only the influence of mutual excitation. The price process is defined in the following.

Definition 4. Mutually excited two-dimensional symmetric Hawkes-based price model

$(N^u(t), \lambda^u(t))$ and $(N^d(t), \lambda^d(t))$ are two pairs of counting and intensity processes of up and down events of mid-price, respectively. Then, the intensity process is formulated so that each is mutual excited with symmetrical exponential kernels, as

$$\lambda^u(t) = \mu + \int_0^t \alpha e^{-\beta(t-s)} dN^d(s) \qquad (9)$$

$$\lambda^d(t) = \mu + \int_0^t \alpha e^{-\beta(t-s)} dN^u(s) \qquad (10)$$

where μ is a baseline intensity, α is mutual excitation, and β is a degree of decay intensity. When the tick size is δ, the price process $S(t)$ is defined, using the difference between counting processes of up and down events of mid-price.

$$S(t) = S(0) + \frac{\delta}{2}\left(N^u(t) - N^d(t)\right) \qquad (11)$$

2.3 Estimation method of transient impact under Hawkes-based pricing model

When we give an external impact on a Hawkes-based price model with an exponential kernel, we analyze the behavior of price process and estimate the decay kernel of transient market impact.

We express the decay of market impact in the Hawkes-based price model by giving the external impact to intensity together with the external market impact to the price process. In the mutual excited two-dimensional Hawkes process, the jump size 1 is immediately excited by α to mutual excitation. Therefore, when externally giving the market impact of $\psi(q)$ to mid-price, the jump size is $2\psi(q)$. It can be reasonably considered that the mutual excitation degree becomes $2\psi(q)\alpha$ according to the jump size.

Therefore, when an external impact occurs at time $t_0 \geq 0$, the impact model of the mutually excited symmetric two-dimensional Hawkes process with an exponential kernel is formulated as follows.

$$\lambda^u(t) = \mu + \int_0^t \alpha e^{-\beta(t-s)} dN^d(s) \qquad (12)$$

$$\lambda^d(t) = \begin{cases} \mu + \int_0^t \alpha e^{-\beta(t-s)} dN^u(s) & (0 \leq t \leq t_0) \\ \mu + \frac{2}{\delta}\psi(q)\alpha e^{-\beta(t-t_0)} + \int_0^t \alpha e^{-\beta(t-s)} dN^u(s) & (t_0 \leq t) \end{cases} \qquad (13)$$

Figure 1 shows the simulation of exerting external impact on the intensity of mutually excited symmetric two-dimensional Hawkes process. Figure 2 shows the 1000 simulation paths of price process.

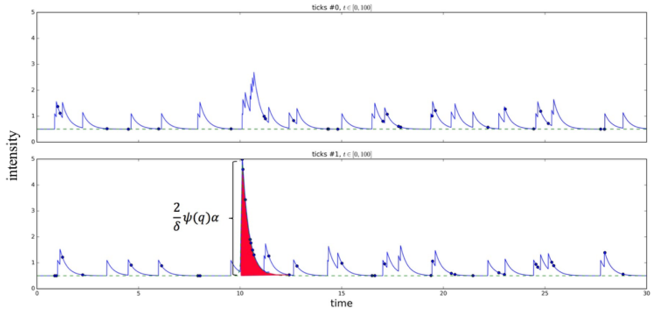

Figure 1: Intensity process of giving an external impact to the Hawkes-based price model at $t = 10$

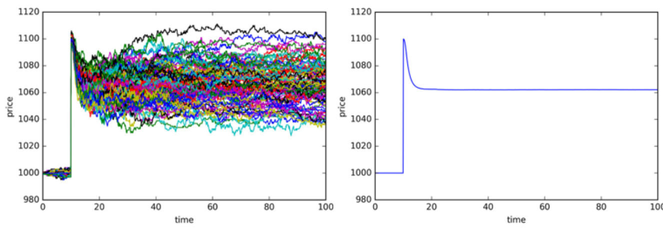

Figure 2: Price process of giving an external impact to the Hawkes-based price model (left: simulation paths, right: average path)

In the case of exponential kernel, the expected value of price process can be explicitly derived in Amaral and

Papanicolaou (2017) (see the proof in appendix). The property of the market impact in this case is summarized as Property 1.

Property 1. Impact property of mutually excited two-dimensional symmetric Hawkes process with exponential kernel

When the temporary impact is any increase function $\psi(q)$, the mutual excitation is α, and the degree of decay intensity is β, the decay kernel of transient impact and permanent impact are determined as follows.

decay kernel of transient impact：$G(t) = e^{-(\alpha+\beta)t}$

permanent impact：$\dfrac{\beta}{\alpha+\beta}\psi(q)$

This is a very simple property, which means that the market impact can be specified only by the mutual excitation α and decay intensity β in the Hawkes process. In fact, the result of the simulation is as shown in Figure 2. When we estimate the mutual excitation α and the intensity attenuation β appropriately from the actual market tick data, we can immediately specify that the decay kernel of the transient impact becomes the exponent of the resilience $\rho = \alpha + \beta$ under the property.

3. EMPIRICAL ANARYSIS

In this section, using the market data, we calibrate the parameters with respect to events of variation of mid-price to a mutually excited two-dimensional symmetric Hawkes process with an exponential kernel and examine some issues.

First, we estimate Hawkes parameters for each stock with different liquidity. Based on the results, we examine how the difference in liquidity affects the estimation result of transient impact. Second, we divide trading time into every 30 minutes over time periods and estimate Hawkes parameters from each divided data. We investigate whether the estimation result of transient impact is affected by intraday seasonality. Finally, we show the case that we fail to estimate the decay kernel using the method of Bouchaud et al. (2004), and we examine the possibility that the method using Hawkes process is more robust.

3.1 Data

We select the stocks in Table 1 from stocks listed on Tokyo Stock Exchange in consideration of liquidity. For all stocks, we use the tick data, which is all of the order history of the market for all 247 business days during the period from January 4, 2012 to December 28, 2012. The tick data only during the trading time (9: 00-11: 30, 12: 30-15: 00) is used.

Table 1: List of stocks for analysis

stock	Average up count	Average down count	liquidity
Softbank(9984)	1709.2	1050.2	high
Kirin Brewery Company(2503)	319.2	231.1	middle
S. T. Corporation (4951)	32.1	23.1	low

3.2 Estimation by the method using Hawkes process

We calibrate the parameters with respect to events of variation of mid-price mutually excited two-dimensional Hawkes process with exponential kernel, and we specify a decay kernel of transient impact for each stock. These results are shown in Table 2. The results show that the higher the liquidity is, the larger the estimated baseline intensity μ is. On the other hand, the mutual excitation α and the decay intensity β are not always large because of high liquidity stock. The result means that the high liquidity stocks have the property that the frequency of occurrences of mid-price variation events is high and the baseline intensity μ is estimated high in the actively traded market, whereas the mutual excitation α and the decay intensity β are estimated independently on the market activity. Therefore, the resilience ρ of transient market impact equal to the sum of α and β does not depend on liquidity.

Table 2: Results of calibration to Hawkes process

	Morning session				Afternoon session			
stock	μ	α	β	$\rho(=\alpha+\beta)$	μ	α	β	$\rho(=\alpha+\beta)$
Softbank(9984)	0.0414	0.0373	0.0799	0.1172	0.0234	0.0403	0.1235	0.1637
Kirin Brewery Company(2503)	0.0106	0.0549	0.1970	0.2520	0.0067	0.0632	0.2725	0.3357
S. T. Corporation (4951)	0.0017	0.0305	0.2478	0.2783	0.0061	0.0646	0.2426	0.3072

3.3 Analysis on intraday seasonality

There is a phenomenon that the high volatility occurs and the large amounts are traded immediately after the opening of the market and just before closing in one day in the stock market. This is called U-shaped pattern, or intraday seasonality. We analyze the influence of the intraday seasonality on resilience of transient market impact by estimating Hawkes parameters from the data of each time periods. The estimates of parameters regarding Softbank (9984) are shown in Figure 3. The baseline intensity μ is large immediately after the opening of the market and gradually decreases over time. In contrast to the baseline intensity μ, the mutual excitation α and the decay intensity β remain unchanged over time. From the result, we confirm that the influence of the intraday seasonality in the Hawkes process is largely depended on only the baseline intensity μ.

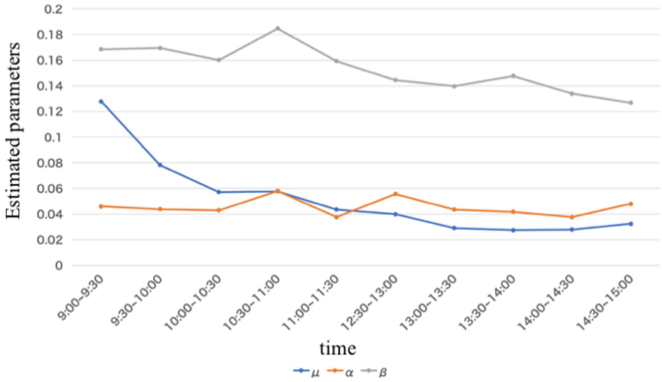

Figure 3: Hawkes parameters of Softbank (9984) in each time periods

3.4 Estimation by the method of Bouchaud et al. (2004)

Bouchaud et al. (2004) assume that stock prices are represented by the sum of transient impacts caused by all past transactions. This supposes that the stock price p_n at time n can be expressed as,

$$p_n = \sum_{-\infty<n'<n} G_0(n-n')\epsilon_{n'} \ln V_{n'} + \sum_{-\infty<n'<n} \eta_{n'} \quad ,(14)$$

where ϵ_n represents a transaction sign at time n, and V_n represents a transacted amount at the time n. It is possible to estimate decay kernel G_0 nonparametrically from market data under the assumption.

Next, we apply the method of Bouchaud et al. (2004) to high and low liquid stocks. These results are shown in Figure 4. Although the decay kernel can be estimated well in high liquid stock, it is difficult to estimate it stably in low liquid stock because of less number of samples.

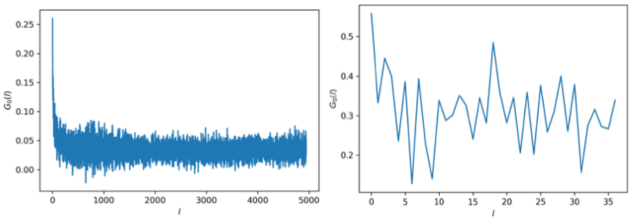

Figure 4: Decay kernel estimation using the method of Bouchaud et al. (2004) (left: Softbank (9984) which is high liquid, right: S. T. Corporation (4951) which is low liquid)

4. CONCLUSION

In this paper, we outline the method of estimating decay kernel of transient impact using a Hawkes process, and conduct the empirical analysis. We find that the resilience ρ of transient market impact does not depend on liquidity and was not affected by the intraday seasonality. Also, we compare the method using a Hawkes process with the method of Bouchaud et al. (2004), and examine the possibility that the method using Hawkes process is more robust. Furthermore, we need to examine the properties of methods using the hypothetical data in order to compare them under the different situations. This is our future research.

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APPENDIX. Analytical solution of decay kernel of transient market impact based on Hawkes-based price model with exponential kernel

The following equations hold by calculating the expected value of probability differential equation followed by intensity of the mutually excited Hawkes process,

$$d\mathbb{E}[\lambda^u(t)] = \beta(\mu - \mathbb{E}[\lambda^u(t)]) + \alpha\mathbb{E}[dN^d(t)]$$
$$d\mathbb{E}[\lambda^d(t)] = \beta(\mu - \mathbb{E}[\lambda^d(t)]) + \alpha\mathbb{E}[dN^u(t)]$$

We calculate the difference using $\mathbb{E}[dN(t)] = \mathbb{E}[\lambda(t)]dt$ due to technical reason for computation. The ordinary differential equation of $\mathbb{E}[\lambda^u(t) - \lambda^d(t)]$ is expressed as,

$$\frac{d}{dt}\mathbb{E}[\lambda^u(t) - \lambda^d(t)] = -(\alpha+\beta)\mathbb{E}[\lambda^u(t) - \lambda^d(t)].$$

The solution is

$$\mathbb{E}[\lambda^u(t) - \lambda^d(t)] = -\left(\lambda^u(t_0) - \lambda^d(t_0)\right)e^{-(t-t_0)(\alpha+\beta)}.$$

Next, we introduce the market impact. When the market impact of $\psi(q)$ occurs at time t_0, intensity of the down event of mid-price is expressed as $\lambda^d(t_0) \to \lambda^d(t_0) + 2\psi(q)\alpha$ caused by price movement of temporary market impact $\psi(q)$, and it can be rewritten as follows.

$$\mathbb{E}[\lambda^u(t) - \lambda^d(t)] = -\Big(\lambda^u(t_0) - \lambda^d(t_0) - \frac{2\alpha}{\delta}\psi(q)\Big)e^{-(t-t_0)(\alpha+\beta)}$$

Therefore, the expected value of price process after the influence of temporary market impact at $t > t_0$ becomes

$$\mathbb{E}[S(t)] = S(t_0) + \psi(q) + \frac{\delta}{2}\mathbb{E}\left[\int_{t_0}^{t}\left(dN^u(s) - dN^d(s)\right)\right]$$

$$= S(t_0) + \psi(q) + \frac{\delta}{2}\int_{t_0}^{t}\mathbb{E}[\lambda^u(t) - \lambda^d(t)]ds$$

$$= S(t_0) + \psi(q) + \frac{\delta}{2}\Big(\lambda^u(t_0) - \lambda^d(t_0) - \frac{2\alpha}{\delta}\psi(q)\Big)\int_{t_0}^{t} e^{-(t-t_0)(\alpha+\beta)}ds$$

$$= S(t_0) + \psi(q) + \frac{\frac{\delta}{2}\left(\lambda^u(t_0) - \lambda^d(t_0)\right) - \alpha\psi(q)\left(1 - e^{-(t-t_0)(\alpha+\beta)}\right)}{\alpha+\beta}.$$

When t→∞, the permanent impact P is

$$P = \lim_{t\to\infty}(\mathbb{E}[S(t)|impacted] - \mathbb{E}[S(t)|no\ impact])$$

$$= \psi(q)\frac{\beta}{\alpha+\beta}$$