Multi-period Optimization Model for Retirement Planning with Private Pension and Life Insurance

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Abstract. In recent years, lots of elderly people worry about their lives after retirement and confer with financial planners. Managing longevity risk is a very important issue for a household in retirement. Private pension and life insurance can be effective tools in managing this risk. Therefore, we need to make an appropriate retirement plan using these products in order not to exhaust retirement savings. There are many studies for retirement planning models in the literatures. Hibiki and Oya(2015) develop a multi-period optimization model for a couple in retirement in the simulated path approach, and derive optimal asset allocation, consumption and annuity using the life table which are modified using the Lee-Carter model and subjective health feeling. The annuity is very important source of income for a retired couple, but the income risk is exposed by the death of a householder or spouse. In this paper, we introduce the life insurance in the model in order to hedge the income risk, and examine the effect. We propose a multi-period stochastic programming model so that financial planners can obtain optimal investment, private pension, life insurance and consumption strategies for a retired couple to manage longevity risk, inflation risk and investment risk and give practical advices. We conduct the numerical analysis to examine the usefulness of the model and show the importance of private pension and life insurance in retirement planning. We conduct the sensitivity analysis to denote the relationship between the individual mortality and the life contingent product.

Keywords: retirement planning, private pension, life insurance, multi-period optimization

1. INTRODUCTION

In recent years, lots of elderly people worry about their lives after retirement and confer with financial planners. Managing longevity risk is a very important issue for a household in retirement. Private pension and life insurance can be effective tools in managing this risk. Therefore, we need to make an appropriate retirement plan using these products in order not to exhaust retirement savings.

There are many studies for retirement planning in the literatures. Gupta and Li (2013) proposed a multiperiod optimization model for longevity risk protection under stochastic lifetime to obtain optimal consumption, investment, and annuitization time, and show the effect of uncertain lifetime on annuity. Hubener, Maurer and Rogalla (2013) derive the optimal demand for stocks, bonds, annuities (single and joint), and term life insurance for a retired couple with uncertainty in both lifetime, using portfolio choice model. As a result, they show that the optimal portfolio is heavily weighted with annuities and life insurance to protect a surviving spouse from loss of annuitized income rather than for bequest. Hibiki and Nishioka (2010) develop a multi-period optimization model for a couple in retirement using the simulated path model, and derive optimal consumption, investment and annuity using the life table modified by subjective health feeling. Hibiki and Oya (2015) extend the model of Hibiki and Nishioka (2010), and solve the problem using the hybrid model which allows conditional (state-dependent) decisions. They also generate the dynamic life table modified using Lee-Carter model and subjective health feeling.

The annuity is very important source of income for a retired couple, but the income risk is exposed by the death of a householder or spouse. In this paper, we introduce the life insurance in the model in order to hedge the income risk, and examine the effect. We propose a multi-period stochastic programming model so that financial planners can obtain optimal investment, private pension, life insurance and consumption strategies for a retired couple to manage longevity risk, inflation risk and investment risk and give practical advices.

2. Modeling

2.1 Structure of problems

We suppose a household is composed of a householder and a spouse who is 65 years old. Planning period is from retired time (65 years) to the time when both die, or maximum period (30 years). One period is one year, and therefore we formulate the 30-period model. Household income is composed of public pension, private pension and life insurance payments, and expenditure is composed of minimum living cost, medical expense, planned consumption, luxury consumption and life insurance premiums. Furthermore, we invest risky assets and cash, and purchase private pension and life insurance at time 0. Private pension is single-premium life annuity, and life insurance is periodic premium term life insurance.

The purpose of retired couple is the case that has a bequest motive or that wants to do luxury consumption. Therefore, the objective function is defined as the sum of the expected additional consumption and the sum of the expected amount of wealth obtained at the time when both couple die or last period minus the expected shortfall from target wealth (to manage longevity risk). The problem is solved so that it can be maximized in the simulated path approach.

2.2 Luxury consumption

2.2.1 Luxury consumption function

We formulate the time-dependent consumption rate to wealth as the piecewise-liner function with several kinked points, as follows.

$$C_{\alpha,t} = \left(\frac{t-\kappa_u}{\kappa_{u+1}-\kappa_u}\right) C_{f,u+1} + \left(\frac{\kappa_{u+1}-t}{\kappa_{u+1}-\kappa_u}\right) C_{f,u} \qquad (1)$$

$$(\kappa_u \le t \le \kappa_{u+1};\, u = 0, \dots, K)$$

$$0 \le k_L \le \frac{C_{f,0}}{C_{f,K+1}} \le k_U \qquad (2)$$

Where K, κ_u, k_L and k_U are the number of the kinked points of the function, the u-th kinked points, the lower and upper multiple bounds of luxury consumption, respectively. Next, we formulate the state-dependent function to wealth, as follows.

$$f^{(i)}(C_{\alpha,t}) = W_t^{(i)} \cdot C_{\alpha,t} \qquad (3)$$

2.2.2 Time preference

Time preference rate is a degree that prefers to consume at present rather than in the future. We need to set it according to requests of the household.

2.3 Survival rate with subjective health feeling

Longevity risk is a very important issue for a retired couple, and therefore we need to estimate survival rate which involves calendar effect explicitly and depends on individuals.

Thus, we use the Lee-Carter method (1992) in order to estimate the dynamic life table involving calendar effect explicitly. Besides, we modify the life table using the subjective health feeling. The subjective health feeling is a subjective evaluation of own health condition. Hibiki and Nishioka (2010) show that the subjective health feeling is related to the survival rate through panel data. The survival rates for male are dependent on subjective health feeling and disease. If a male has no disease, the survival rates are independent on subjective health feeling. On the other hand, if a male has disease, the survival rates are dependent on subjective health feeling. The survival rates for female are dependent on subjective health feeling, regardless of disease.

We show the survival rate modified by the Lee-Carter model for each subjective health feeling in Figure 1.

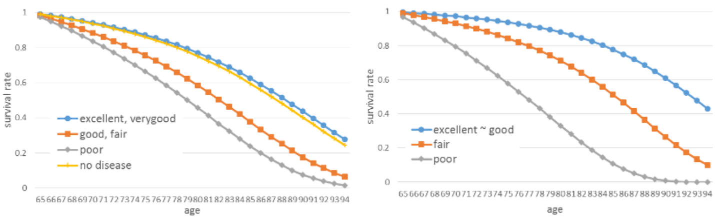

Figure 1: Survival rate based on the dynamic life table and subjective health feeling

2.4 Interest rate model

To model term structures of interest rate, we build upon the Nelson-Siegel model approach in the parameterization of Diebold and Li (2006). This model represents the entire yield curve by only three factors: "level", "slope" and "curvature". The spot rate of maturity τ at time t is formulated as follows.

$$y_t(\tau) = \beta_{1,t} + \beta_{2,t}\left(\frac{1-e^{-\lambda_t \tau}}{\lambda_t \tau}\right) + \beta_{3,t}\left(\frac{1-e^{-\lambda_t \tau}}{\lambda_t \tau} - e^{-\lambda_t \tau}\right) \qquad (4)$$

Where $\beta_{1,t}, \beta_{2,t}$ and $\beta_{3,t}$ denote the value of "level", "slope" and "curvature".

2.5 Hybrid multi-period optimization model

Hibiki (2001) develops the hybrid model in the simulated path approach, which allows conditional decisions to be made for similar states bundled at each time using the sample return generated by the Monte Carlo method. We employ the lattice structures as the modeling structure with respect to the decision nodes in this paper. As example of the lattice structure, we depict the hybrid model on Figure 2.

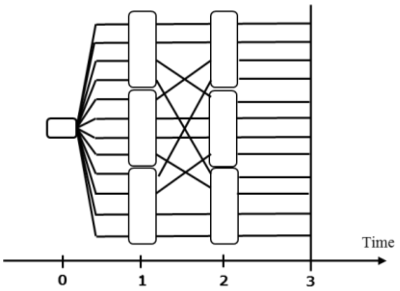

Figure 2: Hybrid model structure

2.6 Investment unit function

An investment unit function is used in the hybrid model in order to express the decision rule which is defined to satisfy the non-anticipativity condition. Let $a_{jt}^{(i)}$ is the investment unit parameter for asset *j*, time *t* and path *i*, and $z_{jt}^{s_t}$ is the criterion variable for asset *j*, time *t* and decision node *s*. Using them, it is defined as Equation (5) which expresses the path-dependent investment unit for path *i*.

$$h^{(i)}\left(z_{jt}^{s_t}\right) = a_{jt}^{(i)} z_{jt}^{s_t} \tag{6}$$

Let $W_t^{(i)}$ denote the amount of wealth of time *t* and path *i*, and $\rho_{jt}^{(i)}$ denote the price of risky asset *j* of time *t* and path *i*. Setting $a_{jt}^{(i)}$, we can deal with various investment rules. Three main settings are shown.

(i) Investment unit decision strategy

$$a_{jt}^{(i)} = 1;\ h^{(i)}\left(z_{jt}^{s_t}\right) = z_{jt}^{s_t} \tag{7}$$

(ii) Investment value decision strategy

$$a_{jt}^{(i)} = \frac{\rho_{j0}}{\rho_{jt}^{(i)}};\ h^{(i)}\left(z_{jt}^{s_t}\right) = \left(\frac{\rho_{j0}}{\rho_{jt}^{(i)}}\right) z_{jt}^{s_t} \tag{8}$$

(iii) Investment proportion decision strategy

$$a_{jt}^{(i)} = \frac{W_t^{(i)}}{\rho_{jt}^{(i)}};\ h^{(i)}\left(z_{jt}^{s_t}\right) = \left(\frac{W_t^{(i)}}{\rho_{jt}^{(i)}}\right) z_{jt}^{s_t} \tag{9}$$

2.7 Formulation

2.7.1 Notation

(1) Sets

S_t : set of decision nodes at time *t*,(*t=1,…,T-1*)

$V_t^{s_t}$: set of paths including node *s* at time *t*, (*t=1,…,T-1*;$s_t \in S_t$)

(2) Parameters

$\tau_{AM,t}^{(i)}, \tau_{AF,t}^{(i)}$: one if householder, spouse is alive at time *t* on path *i* and zero otherwise.

$\tau_{LM,t}^{(i)}, \tau_{LF,t}^{(i)}$: one if householder, spouse dies at time *t* on path *i* and zero otherwise.

$\tau_{A,t}^{(i)}$: one if any family member is alive at time *t* on path *i* and zero otherwise.

$\tau_{L,t}^{(i)}$: one if both family member die at time *t* on path *i* and zero otherwise.

ρ_{j0} : price of risky asset *j* at time 0

$\rho_{jt}^{(i)}$: price of risky asset *j* at time *t* on path *i*

r_0 : Interest rate at time 0,(period 1)

$r_{t-1}^{(i)}$: Interest rate at time *t*-1 on path *i*,(period t)

df_t : discount factor at time *t*

W_0 : Initial wealth

$P_t^{(i)}$: disposable income at time *t* on path *i*

$C_{d,t}^{(i)}$: minimum living cost at time *t* on path *i*

$H_t^{(i)}$: medical expense at time *t* on path *i*

$C_{p,t}^{(i)}$: planned consumption at time *t* on path *i*

k_L, k_U : lower and upper multiple bounds of luxury consumption

A_M, A_F : private pension premiums per unit of a household and a spouse at time 0

a_M, a_F : private pension payments per unit of a household and a spouse at time 0

B_M, B_F : life insurance premiums per unit of a household and a spouse at time *t*,(*t*=0,…,T_L-1), where T_L is insurance period.

θ_M, θ_F : life insurance payments per unit of a household and a spouse at time *t*,(*t*=1,…,T_L)

T_g : guaranteed payment period of private pension

T_L : maturity of life insurance

m : strength of bequest motive

γ : risk aversion coefficient

$\omega_{R,t}$: risk weight coefficient at time *t*, $\left(\sum_{t=1}^{T} \omega_{R,t} = 1\right)$

$\omega_{C,t}$: time preference rate; luxury consumption weight coefficient, $\left(\sum_{t=1}^{T} \omega_{C,t} = 1\right)$

L_v : lower bound of cash

$W_{G,t}^{(i)}$: target wealth at time *t*

(3) Decision variable

D_0^- : cash outflow at time 0

$D_t^{+(i)}, D_t^{-(i)}$: cash inflow, outflow at time *t* on path *i*

$A_t^{+(i)}$: cash inflow of private pension at time *t* on path *i*

$L_t^{+(i)}, L_t^{-(i)}$: cash inflow, outflow of life insurance at time *t* on path *i*

z_{j0} : investment value of risky asset *j* at time 0

$z_{jt}^{s_t}$: investment proportion of risky asset *j* at time *t*, decision node *s*

v_0 : cash at time 0

$v_t^{(i)}$: cash at time *t* on path *i*

$W_t^{(i)}$: wealth at time *t* on path *i*

$C_{\alpha,t}^{(i)}$: luxury consumption value at time *t* on path *i*

$C_{\alpha,t}$: luxury consumption rate at time t

x_M, x_F number of units of private pension for a householder, a spouse at time 0

y_M, y_F : number of units of life insurance for a householder, a spouse at time 0

$q_t^{(i)}$: shortfall from target wealth at time *t* on path *i*

2.7.2 Formulation

The problem is formulated as follow.

$$\text{maximize} \quad m \times \frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T}\tau_{L,t}^{(i)} df_t W_t^{(i)} + (1-m) \times \sum_{i=1}^{I}\sum_{t=1}^{T}\tau_{A,t}^{(i)}\omega_{C,t} df_t C_{\alpha,t}^{(i)} - \gamma\left\{\frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T}\tau_{A,t}^{(i)}\omega_{R,t} df_t q_t^{(i)}\right\} \quad (10)$$

$$\text{subject to} \quad D_0^- = A_M x_m + A_F x_F + B_M y_M + B_F y_F \quad (11)$$

$$A_t^{+(i)} = \begin{cases} a_M x_M + a_F x_F & (t = 1, \dots, T_g) \\ a_M x_M \tau_{A,t}^{(i)} + a_F x_F \tau_{A,t}^{(i)} & (t = T_g + 1, \dots, T) \end{cases} \quad (12)$$

$$L_t^{-(i)} = B_M y_M \tau_{AM,t}^{(i)} + B_F y_F \tau_{AF,t}^{(i)} \quad (t = 1, \dots, T_L - 1) \quad (13)$$

$$L_t^{+(i)} = \theta_M y_M \tau_{LM,t} + \theta_M y_M \tau_{LM,t} \quad (t = 1, \dots, T_L) \quad (14)$$

$$D_t^{-(i)} = L_t^{-(i)} + C_{d,t}^{(i)} + H_t^{(i)} + C_{p,t}^{(i)} + C_{\alpha,t}^{(i)} \quad (15)$$

$$D_t^{+(i)} = A_t^{+(i)} + L_t^{+(i)} + P_t^{(i)} \quad (16)$$

$$D_t^{(i)} = D_t^{+(i)} - D_t^{-(i)} \quad (17)$$

$$\sum_{j=1}^{J}\rho_{j0} z_{j0} + v_0 + D_0^- = W_0 \quad (18)$$

$$W_t^{(i)} = \sum_{j=1}^{J}\rho_{jt}^{(i)} h^{(i)}\left(z_{j,t-1}^{s_t}\right)\tau_{A,t}^{(i)} + \left(1 + r_{t-1}^{(i)}\right)v_{t-1}^{(i)} + D_t^{(i)} \quad (19)$$

$$= \sum_{j=1}^{J}\rho_{jt}^{(i)} h^{(i)}\left(z_{j,t}^{s_t}\right)\tau_{A,t}^{(i)} + v_t^{(i)} \quad (t < T_R)$$

$$W_t^{(i)} = \left(1 + r_{t-1}^{(i)}\right)W_{t-1}^{(i)} + D_t^{(i)} \quad (t > T_R) \quad (21)$$

$$W_t^{(i)} + q_t^{(i)} \geq W_{G,t}^{(i)}\ ;\ q_t^{(i)} \geq 0 \quad (22)$$

$$\mathrm{z_{j0}}, z_{jt} \geq 0\ ;\ \mathrm{v}_t^{(i)} \geq 0 \quad (23)$$

$$\sum_{j=1}^{J}\rho_{j0} z_{j0} \leq (1 - L_v)(W_0 - D_0^-) \quad (24)$$

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\rho_{jt}^{(i)} h^{(i)}\left(z_{j,t-1}^{s_t}\right) \leq (1 - L_v)\sum_{i=1}^{I} W_t^{(i)} \quad (25)$$

$$W_T^{(i)} \geq 0 \quad (25)$$

$$0 \leq x_M \leq 1\ , 0 \leq x_F \leq 1 \quad (27)$$

$$0 \leq y_M\ , 0 \leq y_F \quad (28)$$

3. Numerical analysis

We conduct the numerical analysis for a hypothetical household. The base parameters are in Table 1.

All of the problems are solved using Numerical Optimizer (Ver. 18.1) – mathematical programming software package developed by NTT DATA Mathematical System, Inc. on Windows 7 personal computer which has Xeon E5-1603 2.80GHz CPU and 64GB memory.

Table 1: Base parameters

Parameters	values
initial wealth	$W_0 = 2{,}000$
expected rate of return of stock	$\mu_s = 5\%$
standard deviation of rate of return of stock	$\sigma_s = 20\%$
subjective health feeling (householder)	"poor"
subjective health feeling (spouse)	"excellent"
bequest motive	$m = 0$
planning period, investment period	$T = 30$ $T_R = 10$
lower bounds of cash	$L_v = 10\%$
private pension premium per unit	$A_M = 1{,}694$ $A_F = 2{,}093$
private pension payment per unit	$a_M, a_F = 90$
guaranteed payment period of private pension	$T_g = 10$
life insurance premium per unit	$B_M = 27.116$ $B_F = 12.4464$
life insurance payment per unit	$\theta_M, \theta_F = 1000$
maturity of life insurance	$T_L = 15$
number of kinked points of consumption function	$K = 1$
kinked point	$\kappa_1 = 15$
Lower and upper multiple bounds of luxury consumption	$k_U = 3, k_L = 1$
risk aversion coefficient	$\gamma = 1$
risk weight coefficient	$\omega_{R,t} = 1/T$
inflation rate	$f = 0\%$
number of paths	$I = 10{,}000$
number of decision nodes	5

3.1 Effect of life insurance

We conduct the numerical analysis for a hypothetical household to examine the usefulness of the model and show the importance of private pension and life insurance in retirement planning. Therefore, we solve the problem in the case with and without life insurance. Figure 3 shows the efficient frontier and luxury consumption rate. The efficient frontier shifts to upper left and luxury consumption rate increases by considering life insurance. The reason is that both a householder and a spouse hedge the income risk of public and private pensions with early death by purchasing life insurance. Consequently, the household with life

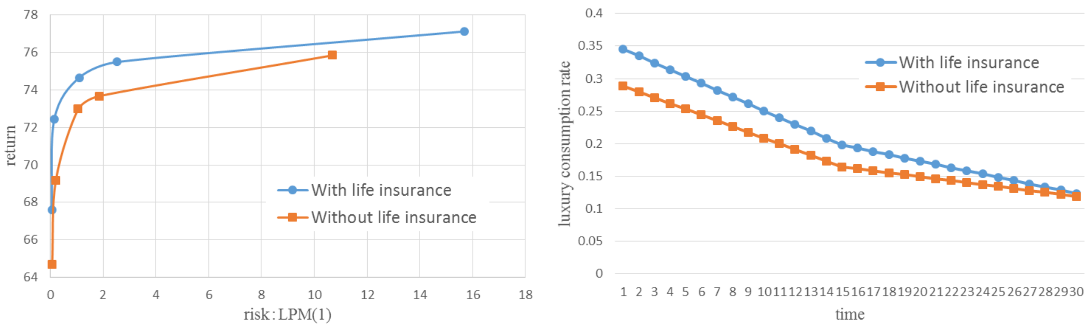

Figure 3: Efficient frontier and luxury consumption rate

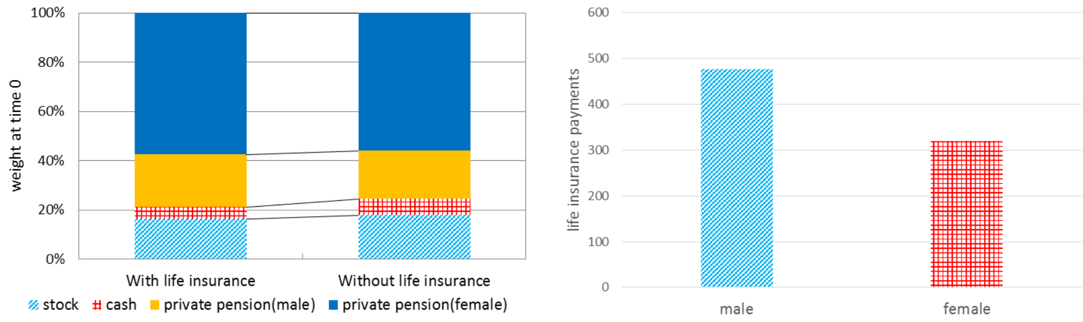

Figure 4: Asset allocation at time 0 and life insurance payments

insurance spend more money for luxury consumption than the household without life insurance because of avoiding shortfall risks.

Figure 4 shows asset allocation at time 0 and life insurance payments. The household with life insurance purchases more private pension than the household without life insurance. In contrast, the household without life insurance purchases more liquid assets than the household with life insurance. The reason is that as mentioned above, the household can hedge the income risk of public and private pensions with early death by purchasing life insurance. Therefore, even although the subjective health feeling of spouse is "excellent", a spouse can increase the fraction of private pension by purchasing the amount of life insurance.

As a result, the case with life insurance can stabilize the future income and can hedge longevity risk. On the other hand, the case without life insurance decreases the fraction of private pension in order to avoid income risk with early death.

3.2 Sensitivity analysis of mortality rate

We conduct the sensitivity analysis of mortality rate to examine the relationship between the individual mortality and the life contingent product. We solve six combinations of subjective health feeling of family in Table 2.

Table 2: combinations of subjective health feeling

spouse / householder	excellent ~good(ex)	poor (po)
excellent, very good(ex)	exex	expo
good, fair(go)	goex	gopo
poor(po)	poex	popo

Figure 5 shows the objective function value and asset allocation at time 0. The better the subjective health feeling is, the larger the objective function value is. As in section 3.1, we can hedge the income and longevity risk by purchasing private pension and life insurance. Therefore, the higher the survival rate is, the more the household can consume. The subjective health feeling affects the amount of private pension. In order to analyze the relationship between the survival rate and the demand of private pension and life insurance, the present value (discount rate 0.5%) of the probability that only either one is alive is defined as "difference in survival rate", and the ratio of the amount of private pension of having a high survival rate to the initial wealth is defined as "private pension ratio". Figure 6 shows the relationship between difference in survival rate and private pension ratio and relationship between private pension premiums and life insurance payments. There is a high correlation (correlation coefficient is 0.947) between them. The higher the probability that only either one is alive is, the more amount of private pension of having a high survival rate the household purchase. Similarly, there is a high correlation (correlation coefficient is 0.972) between private pension premiums and life insurance payments for female. On the other hand, there is a lower correlation (correlation coefficient is 0.761) for male than female. The purpose of purchasing life insurance for female is to hedge the income risk of private pension. The purpose for male is to hedge the income risk of private and public pensions, because a spouse cannot receive public pension if a householder dies. Consequently, own subjective health feeling affects the demand of life insurance in addition to demand of private pension.

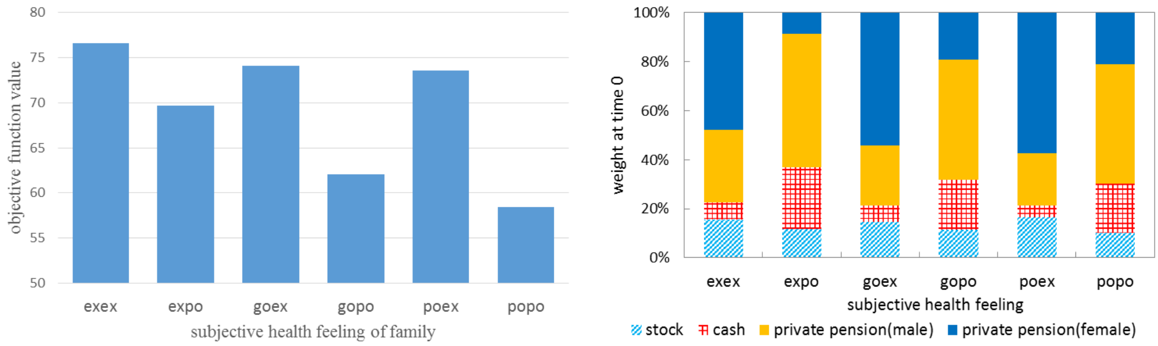

Figure 5: Objective function value and asset allocation at time0

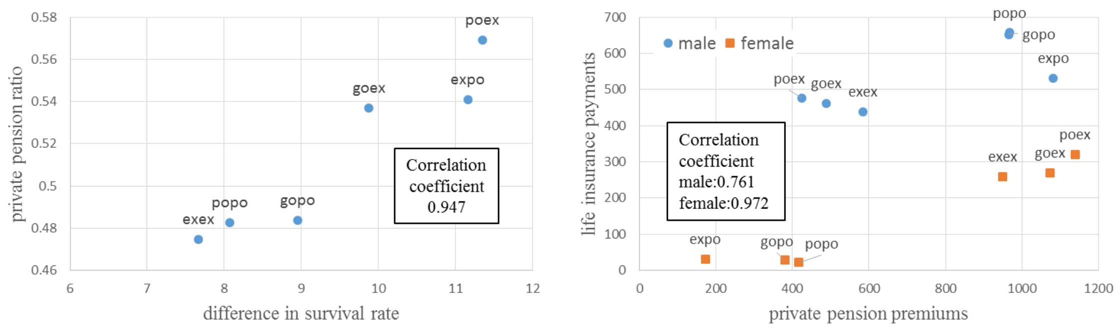

Figure 6: Relationship between difference in survival rate and private pension ratio and relationship between private pension premiums and life insurance payments

4. Conclusion

In this paper, we propose a multi-period stochastic programming model for a retired couple to manage longevity risk, inflation risk and investment risk. We conduct the numerical analysis for a hypothetical household to examine the usefulness of the model and show the importance of private pension and life insurance in retirement planning. We conduct the sensitivity analysis of mortality rate to examine the relationship between the individual mortality and the life contingent product.

In the future research, we need to construct the integrated model with life planning model in the working generation.

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