Dynamic Optimal Execution Models with Transient Market Impact and Downside risk

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Abstract. When institutional investors trade a large amount of a stock in the market, the trading amount might impact the price, and the price change is called market impact. In addition, their trading is always exposed to uncertain price change, which is called timing risk. They need to evaluate quantitatively market impact and timing risk, and decide optimal execution strategy in consideration of the trade-off between them. Many previous studies assume temporary / permanent market impact, but recently some studies are discussed under the assumption of transient market impact. On the other hand, institutional investors need to manage the downside risk when they execute the order to meet the trading needs within the target cost. In our paper, we discuss the dynamic optimization models with transient market impact and downside risk in order to decide the optimal execution strategy. Specifically, we propose the following three types of the models, based on Takenobu and Hibiki (2016) which assume temporary / permanent market impact.

(1) Hybrid multi-period stochastic optimization model using Monte Carlo simulation.
(2) Piecewise liner approximation model based on the hybrid model
(3) Iterative model with static execution strategy

We solve the optimal execution problems using the models, and conduct the sensitivity analysis in order to examine the usefulness of the models. In addition, we compare three models, and evaluate the characteristics and the difference among them. We estimate the market impact function and other parameters using market data, and derive the optimal execution strategies for practical use.

Keywords: dynamic optimal execution, transient market impact, market order, downside risk

1. INTRODUCTION

When institutional investors trade a large amount of a stock in the market, the trading amount might impact the price, and the price change is called market impact. In addition, their trading is always exposed to uncertain price change, which is called timing risk. They need to evaluate quantitatively market impact and timing risk, and decide optimal execution strategy in consideration of the trade-off between them. Many previous studies assume temporary / permanent market impact. Bertsimas and Lo (1998) derive the optimal strategy of minimizing expected cost which assumes implicitly an investor is risk neutral. Almgren and Chriss (2001) derive the static optimal strategy using the variance of cost as market timing risk measure. They use the variance of total cost as a risk measure. On the other hand, institutional investors need to manage the downside risk when they execute the order to meet the trading needs within the target cost. In addition, the investors develop the dynamic execution strategy in consideration of the price impact cost and market timing risk appropriately. Takenobu and Hibiki (2016) formulate dynamic optimal execution models with the first-order lower partial moment (LPM) as a downside risk measure.

Recently, Bouchaud *et al.* (2006) show that the price

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impact is transient in the real market. In addition, some studies are addressed under the assumption of transient market impact. Gatheral *et al.* (2011) derive the optimal strategy of minimizing expected cost. Alfonsi *et al.* (2012) derive the static optimal strategy in consideration of the variance of cost, which is the same problem setting as Almgren and Chriss (2001). Lorenz and Schied (2013) derive the dynamic optimal strategy. Their admissible strategy is the sum of a sell and a buy strategy. In contrast, our admissible execution strategy is a pure sell strategy in order to meet the execution needs of institutional investors who get a contract to sell stocks under a target cost.

In our paper, we discuss the dynamic optimization models with transient impact and downside risk in order to decide the optimal execution strategy and we formulate the optimal execution problem in a discrete time. Specifically, we propose the following three types of the models, based on Takenobu and Hibiki (2016).

(1) Hybrid multi-period stochastic optimization model with dynamic execution strategy using Monte Carlo simulation (called hybrid model hereafter).
(2) Piecewise liner model which is an extended model of the hybrid model (called piecewise model hereafter).
(3) Iterative model with static execution strategy

We can show the differences among them as in Table 1, and evaluate the characteristics. We conduct the numerical analysis, and examine the usefulness of the models.

Table 1: Comparison with three models

Model	Hybrid	Piecewise	Iterative
Conditional decision	○	○	○
Flexibility of decision making	△	○	○
Inclusion of practical constraint	○	○	×
Computation load	very high	high	low
Constraint of price process	×	△	○

2. THE OPTIMAL EXECUTION PROBLEM

We set up the problems with reference to Gatheral *et al.* (2011) and Alfonsi *et al.* (2012). We assume that we hold a block of shares X of a single security which initial price is P_0. We needs to sell a stock in the market by time horizon T. We divide T into K intervals of length $\tau = T/K$. We plan to hold x_k shares at time k $(k = 1,..,K)$, and therefore we shall sell $x_{k-1} - x_k$ between $k-1$ and k. Average rate of trading during period k is $\mathrm{v_k} = (x_{k-1} - x_k)/\tau$ $(\tau = T/K)$.

2.1 MARKET IMPACT MODEL

Many previous studies have used a temporary / permanent market impact model. In contrast, we use a transient market impact model (Figure 1). We define transient market impact as follow. When τv_u shares are executed at time $u-1$, the temporary market impact $\tau h_0 v_u$ occurs, and then at time k, the temporary market impact decays to $G_{k-u+1}(\tau h_0 v_u)$. The decay kernels of the form, $G: [0,\infty) \to [0,1]$, is defined as follows, with reference to Alfonsi *et al.* (2012). We assume G as an exponential or a power function.

$$G_{k-u+1} = exp(-\rho_e(k-u+1)\tau) \qquad (\rho_e \geq 0) \qquad (1)$$

$$G_{k-u+1} = (1+\lambda(k-u+1)\tau)^{-\rho_p} \qquad (\rho_p, \lambda \geq 0) \qquad (2)$$

Therefore, the market impact at time k derived from execution at time u − 1 can be formulated as follow.

$$MI\big((k-u+1)\tau\big) = \tau h_0 v_u G_{k-u+1} \qquad (3)$$

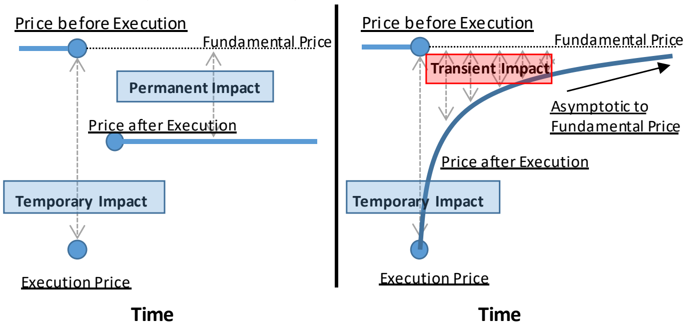


Figure 1: temporary / permanent and transient market impact

2.2 PRICE DYNAMICS

We assume that price process follows the arithmetic Brownian motion. So, the evolution of the fundamental price P_k and execution price $\tilde{P}_k$ considering market impact can be formulation as follow.

$$P_k = P_0 + \sigma\sqrt{\tau}\sum_{u=1}^{k}\xi_u - \tau\sum_{u=1}^{k} h_0 v_u G_{k-u+1} \qquad (4)$$

$$\tilde{P}_k = P_{k-1} - \tau h_0 v_k \qquad (5)$$

We represent the random price change as $\sigma\sqrt{\tau}\xi_u$ using daily standard deviation, σ, and uncertain fluctuations in period $[\mathrm{u}-1, \mathrm{u}]$, $\xi_\mathrm{u} \sim \mathrm{N}(0,1)$.

2.3 DEFINITION OF EXECUTION COST

We evaluate the total cost of trading, or implementation shortfall, for selling the amount of security which is the difference between the initial market value and the final capture of the trade derived using trading policy. It is expressed as $C_K = XP_0 - \sum_{k=1}^{K}(x_{k-1} - x_k)\tilde{P}_k$, which is nondimensionalized by dividing by $\sigma\sqrt{T}X$, like Lorenz and Almgren (2011).

$$\hat{C}_K = \mu_h K \sum_{k=1}^{K}\sum_{u=1}^{k} G_{k-u}(\hat{x}_{k-1} - \hat{x}_k)(\hat{x}_{u-1} - \hat{x}_u) - \sigma\sqrt{\tau}\sum_{k=1}^{K-1}\xi_k \hat{x}_k \qquad (6)$$

$$\mu_h = (h_0 X/K)/(\sigma\sqrt{T}) \qquad (7)$$

where, $\hat{C}_K = (1/\sigma\sqrt{T}X)C_K$, $\hat{x}_k = (1/X)x_k$, and μ_h is called "market power" which is a non-dimensional preference-free measure of portfolio size in terms of its ability to move the market, identified by Almgren and Lorenz (2007). The second and third terms of Equation (6) show the market impact cost, and timing risk, respectively. Hereafter, we remove caret for simplicity

2.4 STATE-INDEPENDENT MODEL (N1 MODEL)

We formulate state-independent model with downside risk (called N1 model) with reference to Alfonsi *et al.* (2012). We use the first-order lower partial moment (LPM) as risk measure which is expected value of total cost C_K beyond target cost C_G. The LPM can be formulated using expected cost $\bar{C}_K$ and variance of cost σ_C^2, as follows.

$$LPM(\bar{C}_K) = \int_{C_G}^{\infty} (C_K - C_G) f(C_K) dC_K$$

$$= \sigma_C \phi(Q) + \sigma_C Q \Phi(Q) \quad (8)$$

$$\bar{C}_K = \mu_h K \sum_{k=1}^{K} \sum_{u=1}^{k} G_{k-u}(x_{k-1} - x_k)(x_{u-1} - x_u) \quad (9)$$

$$\sigma_C = \frac{1}{K} \sum_{k=1}^{K-1} x_K^2 \quad (10)$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ represent density function and cumulative distribution function of standard normal distribution respectively, and $Q = (\bar{C}_K - C_G)/\sigma_C$.

We formulate the N1 model minimizing the objective function which is the expected total cost plus LPM multiplied by risk aversion γ as follows.

$$\text{minimize} \quad \bar{C}_K + \gamma \cdot LPM(\bar{C}_K) \quad (11)$$

$$\text{subject to} \quad 1 \geq x_1 \geq x_2 \geq \cdots \geq x_{K-1} \geq 0 \quad (12)$$

3. DYNAMIC OPTIMAL EXECUTION MODELS

We propose three kinds of models; hybrid model, piecewise model, and iterative model.

3.1 HYBRID MODEL

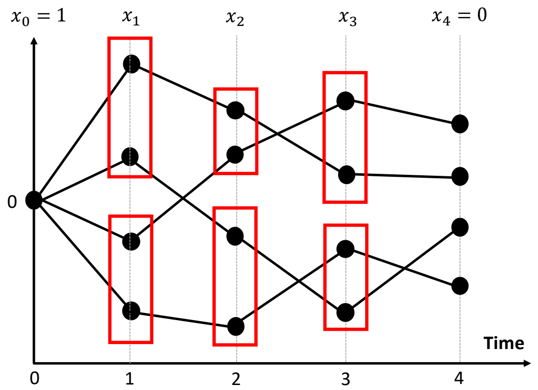


Figure 2: Hybrid model structures

The hybrid model allows conditional decisions to be made for similar states bundled at each time using samples returns generated by the Monte Carlo method, suggested by Hibiki (2006). We bundle samples according to the total cost, and the same decisions are made in similar states. As a sample of hybrid model, we depict the N2 model of four periods which have two nodes and four sample paths in Figure 2.

3.1.1 FORMULATION

We formulate hybrid NS (S is the number of nodes) model for the optimal execution problem as follow.

(1) Notations

a) Parameters

J: number of sample paths $(j = 1, \dots, J)$

K: number of periods $(k = 1, \dots, K)$

S: number of nodes $(s = 1, \dots, S)$

$\xi_k^{(j)}$: random price change on path j at time k.

μ_h: market power

γ: risk aversion coefficient

C_G: target cost

b) Variables

x_1: residual fraction of order held at time 1, determine d at time 0

y_k^s: residual fraction of order held on node s at time k

$q^{(j)}$: deviation of total cost C_K beyond C_G on path j

$x_k^{(j)}$: residual fraction of order held on path j at time k

$C_k^{(j)}$: cumulative cost on path j up to time k

$LPM(C_K)$: first-order lower partial moment of total cost

(2) Formulation

$$\text{Minimize} \quad \frac{1}{J} \sum_{j=1}^{J} C_K^{(j)} + \gamma LPM(C_K) \quad (13)$$

subject to

$$C_k^{(j)} = C_{k-1}^{(j)} + \mu_h K \sum_{u=1}^{k} G_{k-u}(x_{k-1}^{(j)} - x_k^{(j)})(x_{u-1}^{(j)} - x_u^{(j)})$$

$$-x_k^{(j)} \xi_k^{(j)} / \sqrt{K} \quad (14)$$

$$(C_0^{(j)} = 0, x_0^{(j)} = 1, x_1^{(j)} = x_1, x_K^{(j)} = 0)$$

$$C_K^{(j)} - q^{(j)} \leq C_G \quad (15)$$

$$LPM(C_K) = \frac{1}{J} \sum_{j=1}^{J} q^{(j)} \quad (16)$$

$$q^{(j)} \geq 0 \quad (17)$$

$$x_k^{(j)} \leq x_{k-1}^{(j)} \quad (18)$$

$$x_{K-1}^{(j)} \geq 0 \quad (19)$$

$$x_k^{(j)} = \begin{cases} y_k^1 & \left(C_{k-1}^{(j)} \leq \theta_{k-1}^1\right) \\ y_k^s & \left(\theta_{k-1}^{s-1} \leq C_{k-1}^{(j)} \leq \theta_{k-1}^s\right) \\ y_k^S & \left(C_{k-1}^{(j)} \geq \theta_{k-1}^{S-1}\right) \end{cases} \quad (20)$$

Equation (14) is the calculation of total cost at time k.

Equations (15) to (17) are used for the calculation of LPM. Equations (18) and (19) are the constraints of non-increasing in time for residual fractions. Equation (20) shows the residual fraction of order $x_k^{(j)}$, which is the step function of cumulative cost $\mathrm{C}_{k-1}^{(j)}$, and the residual fraction of order on node s, y_k^s, are determined. θ_{k-1}^s are portioned points of $\mathrm{C}_{k-1}^{(j)}$ on node s. Conditional decisions are allowed to be made in the model.

3.1.2 OPTIMAL EXECUTION STRATEGY AND STEP FUNCTION

We estimate a step function through numerical analysis of hybrid model with many nodes. We set the following parameters; $\mathrm{J} = 50{,}000$, $\mathrm{K} = 6$, $\mathrm{S} = 25$, $\mathrm{C_G} = 0.3$. We assume transient market impact as exponential function, $\rho_e = 0.3$, (Equation (1)). We derive dynamic optimal execution strategy using N25 model for four cases (Table 2).

Table 2: Parameters for four cases

Parameter	Basic case	Case 1	Case 2	Case 3
γ	1	10	1	1
μ_h	0.1	0.1	0.2	0.1
C_G	0.3	0.3	0.3	0.05

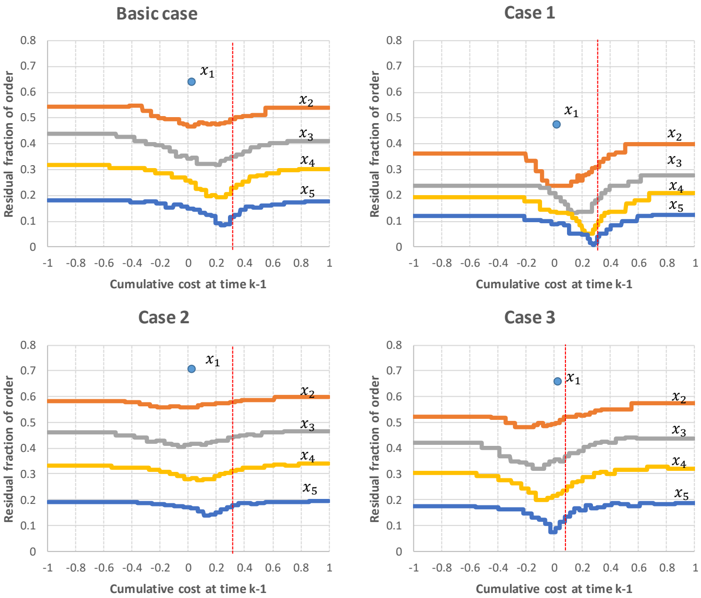


Figure 3: Dynamic optimal execution strategy in each case

We show dynamic optimal execution strategy in Figure 3. The optimal residual orders are state-dependent, and become almost short-butterfly forms which consist of V-shaped part and flatter parts, with respect to the cumulative cost. The result is consistent with Takenobu and Hibiki(2016) even in consideration of transient market impact. As the cumulative cost becomes close to the kinked point, the chance of risk becomes large. Therefore, market impact is tolerated and the amount of residual order becomes small in order to avoid the increase in timing risk which is difficult to control. On the other hand, the residual order becomes large in order to have a chance of reducing the total cost by the rise in the stock price as the cumulative cost becomes larger than the kinked point. We need the large number of nodes in order to express the short-butterfly form using the step function in the hybrid model. This leads to the large-scale optimization problem and the increase in the computation time. Therefore, we propose the piecewise linear model with transient market impact as well as Takenobu and Hibiki (2016). It is expected that the computation time is reduced due to the piecewise linear approximation. In addition, we also propose to solve the problem with a static model iteratively to reduce the computation time drastically.

3.2 PIECEWISE LINEAR MODEL

Cumulative cost to the minimum fraction needs to be given in the piecewise linear model before solving the problem. But, it is difficult to determine it because we cannot find it without solving the problem using the hybrid model. In our paper, we determine it as $\mathrm{C_G} - A_k - B_k$, "target cost minus the sum of expected cumulative cost after time k and risk adjusted term" derived by using the one-period analytical model formulated under the limited assumption.

$$A_k = \bar{C}_K - \bar{C}_{k-1} \tag{21}$$

$$B_k = b_k x_k^{min} u^* \tag{22}$$

$$b_k = \frac{1}{x_k^{min}} \sqrt{\frac{1}{K} \sum_{d=k}^{K-1} \left(x_d^{min}\right)^2} \tag{23}$$

$$u^* = \frac{\phi(u^*)}{1 + 1/\gamma - \Phi(u^*)} \tag{24}$$

where x_k^{min} is residual fraction of order in the kinked point. Due to space limitation, we omit the explanation of the analytical model.

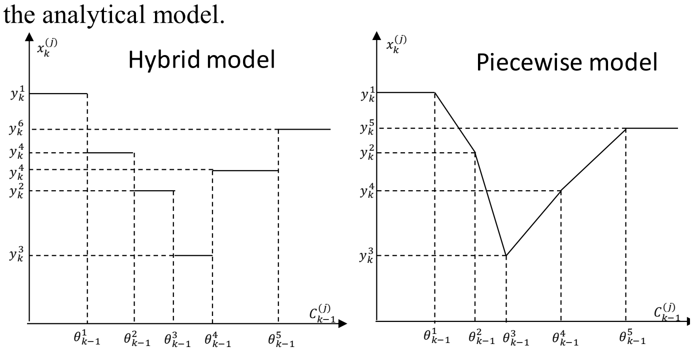


Figure 4: Extension to piecewise model

We formulate the piecewise model (PS model, S is the number of ranges) which is extended from the hybrid model. We also use the parameters and variables defined section 3.1.1. The formulation of the piecewise model is basically the same as that of the hybrid model except the calculation of $x_k^{(j)}$ as follows.

$$x_k^{(j)} = \begin{cases} y_k^1 & \left(C_{k-1}^{(j)} \le \theta_{k-1}^1\right) \\ \left(1-\alpha_k^{(j)}\right) y_k^{s-1} + \alpha_k^{(j)} y_k^s & \left(\theta_{k-1}^{s-1} \le C_{k-1}^{(j)} \le \theta_{k-1}^s\right) \\ y_k^S & \left(C_{k-1}^{(j)} \ge \theta_{k-1}^{S-1}\right) \end{cases} \quad (25)$$

$$\alpha_k^{(j)} = \frac{C_{k-1}^{(j)} - \theta_{k-1}^{s-1}}{\theta_{k-1}^s - \theta_{k-1}^{s-1}} \quad (26)$$

We can decide the optimal residual orders smoothly according to Equation (25). It is expected that the similar optimal execution strategy can be derived using the piecewise linear model with a smaller number of ranges than the hybrid model.

3.3 ITERATIVE MODEL

We formulate the iterative model to reduce the computation time. It can derive the state-dependent decisions approximately to the piecewise model.

Consider the problem of solving for x_k. Given C_{k-1} and x_{k-1}, the expected total cost $\bar{C}_K$ and variance of total cost are formulated using residual orders after time k, x_d $(d = k+1, \dots, K-1)$, as follows.

$$\bar{C}_K = C_{k-1} + \mu_h K \sum_{d=k}^{K} \sum_{u=1}^{d} G_{k-u}(x_{d-1} - x_d)(x_{u-1} - x_u) \quad (27)$$

$$\sigma_C^2 = \frac{1}{K} \sum_{d=k}^{K-1} x_d^2 \quad (28)$$

LPM is calculated by Equation (8), and we derive x_k by minimizing the objective function, $f(x_k) = \bar{C}_K + \gamma LPM(\bar{C}_K)$, as follows.

$$\frac{a_1}{a_2} - x_k = \frac{b_k h(u)}{2\mu_h K a_2}\ , \quad (29)$$

where u is expressed by C_{k-1} and x_k. In addition, a_1 and a_2 can be expressed by x_{k-1}, but they are independent of x_k. Equation (29) cannot be solved analytically, but we obtain x_k numerically. Specifically, we give the optimal residual order of time $k-1$ derived by the N1 model as the parameter x_{k-1}, and obtain x_k satisfying the Equation (29). The state-dependent residual orders can be derived approximately by implementing the procedure iteratively through the planning period.

4. NUMERICAL ANALYSIS

We derive optimal execution strategy with piecewise model and iterative model using hypothetical data in order to compare the results of hybrid model, and conduct the sensitivity analysis in order to examine the usefulness of the models. All of the problems are solved using Numerical Optimizer (Ver 18.1) — mathematical programming software package developed by NTT DATA Mathematical System, Inc. on Windows 10 personal computer which has Corei7-6700K, 4.00GHz CPU and 32GB memory.

4.1 SETTING

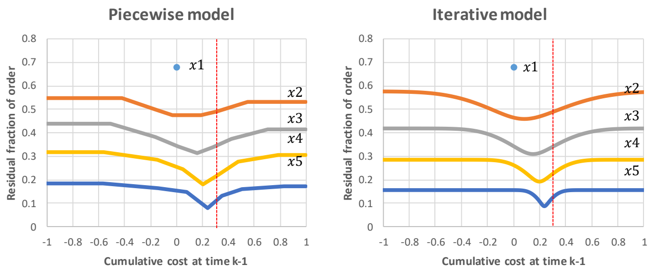
(1) Parameters of basic case.

$K = 6, J = 50{,}000, C_G = 0.3, \gamma = 1, \mu_h = 0.1, \rho_e = 3.$

(2) How to classify paths

In the hybrid and piecewise models, we set eight kinds of S (=2,4,6,8,12,16,20,24) and the same number of ranges at each time. In the hybrid model, the same number of paths included in each range (J/S). In the piecewise model, the number of division is symmetric about the kinked point which is the cumulative cost to the minimum fraction $(C_G - A_k - B_k)$ estimated in section 3.2.

4.2 BASIC ANALYSIS



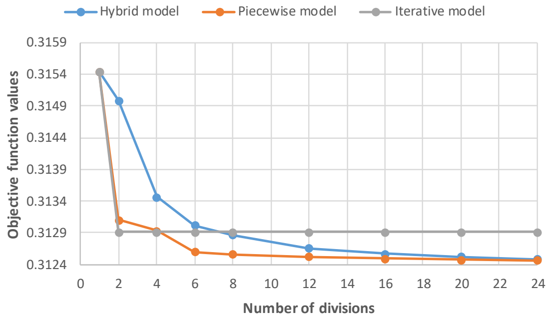


Figure 6: Objective function values for different number of divisions

We show optimal execution strategies on Figure 5 derived by piecewise model and iterative model in the basic case. In comparison with basic case in Figure 3, similar strategies can be derived. In addition, we show the objective function values for different number of divisions in Figure 6. When $S = 1$, hybrid model and piecewise model is equivalent to the N1 model, and since iterative model determines the residual orders for each path, we describe fixed value. Comparing N16 and P6 models or N24 and P8 models which have the similar objective function values, the computation time can be reduced to about 30%.

4.3 SENSITIVITY ANALYSIS

In order to examine whether piecewise model and iterative model can derive results similar to hybrid model even when the market environment changes, we depict the error from the objective function value derived by N24 model for different parameters of μ_h, γ, or ρ_e in Figure 7.

We find that piecewise model and iterative model are close to the hybrid model in any market condition. Furthermore, the piecewise model is more accurate than the iterative model and robust against changes in the market condition.

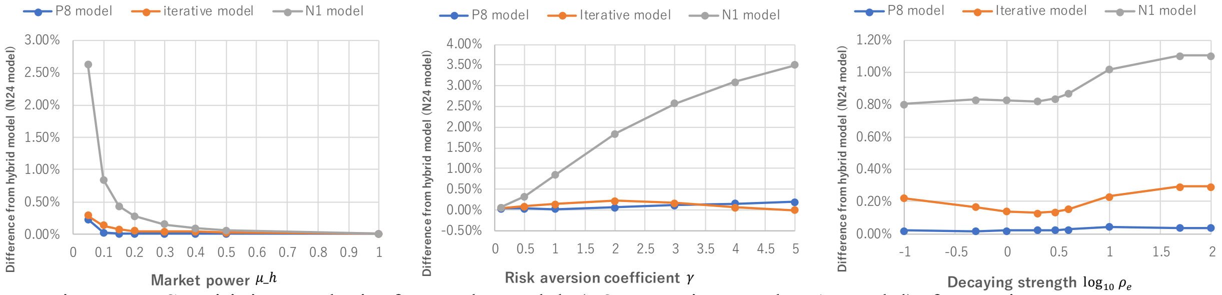


Figure 7: Sensitivity analysis for each model (P8, Iterative, and N1 model) for various μ_h, γ, or ρ_e

5. ANALYSIS OF MARKET DATA

We estimate the market impact function and other parameters using market data, and derive the optimal execution strategies for practical use. Especially, we estimate temporary market coefficient h_0, daily standard deviation σ in order to calculation market power μ_h, and transient market impact function G. We use the 2012 tick data to estimate the parameters. We estimate σ and h_0 as "Realized Volatility" and "spread over best bid quality" before day of execution. G is estimated with reference to Bouchaud *et al.* (2006).

Table 3: estimated parameters using market data

Parameter	Softbank (9984)	Docomo (9437)
P_0	3140 yen	130,000 yen
σ	26.68 yen	624 yen
h_0	2.5×10^{-5} yen	0.23 yen
G_d	$(1 + 72.11d \cdot \tau)^{-0.32}$	$(1 + 1.76d \cdot \tau)^{-0.09}$

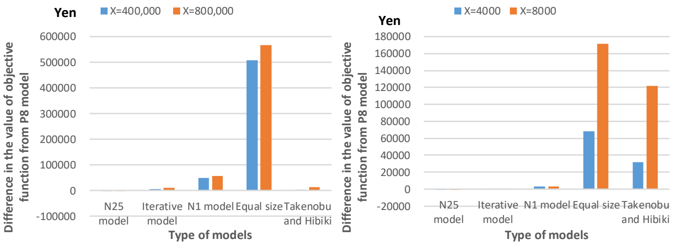


Figure 8: Difference of objective function values from P8 model for each execution strategy

Softbank (9984) and Docomo (9437), which are largescale stocks listed with first section of the Tokyo Stock Exchange, are supposed to be executed and the estimated parameters are shown in the Table 3. Decay speed of transient market impact of Softbank is fast and Docomo is slow. We derive optimal execution strategy using the proposed models (N25 model, P8 model, Iterative model) and comparative model (Execution strategy of trading in equal size lots, N1 model, piecewise model with temporary / permanent market impact proposed by Takenobu and

Hibiki (2016)) and compare the objective function values with that of P8 model in Figure 8. The objective function values of the proposed three kinds models are smaller than the others (N1 model, trading in equal size, and Takenobu and Hibiki model). Therefore, we find the proposed models are useful in practice.

6. CONCLUSION

We propose different three types of models, hybrid model, piecewise model, and iterative model, as dynamic optimal execution model, and show the characteristics and usefulness of each model through the state-dependent strategy using hypothetical data and real market data.

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