Asset Allocation Model with Tail Risk Parity

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Abstract. Asset allocation strategy is important to manage assets effectively. In recent years, risk parity strategy attracts attention in place of traditional mean-variance approach. Risk parity portfolio is one of the risk-based portfolios, and it equalizes risk contributions across all assets included in the portfolio. Specifically, the equally-weighted risk contribution is calculated by decomposing the standard deviation of the portfolio's return. In addition, some studies propose the tail risk parity strategy which equalizes the risk contribution of downside risk measure (Alankar et al., 2012, Boudt et al., 2013), and use conditional value-at-risk (CVaR) as a risk measure. In this paper, we first compare tail risk parity strategies with CVaRs estimated by three kinds of estimation methods (Delta-normal method, historical-simulation method, and Monte Carlo method), and examine the characteristics of the risk parity portfolios. We also implement the backtest for eighteen years using the historical data of Nikkei 225, Citi JPGBI (Japan government bond index), S&P500, and Citi USGBI. We find the estimated expect return and distribution affect optimal investment ratios and portfolio's performance, but mutual dependence between assets does not affect them.

Keywords: finance, asset allocation, risk parity, risk budget, down side risk

1. Introduction

Asset allocation strategy is important to manage assets effectively. The standard asset allocation model is the mean-variance model. This model uses expected returns, standard deviations and correlations of assets, optimal portfolio is uniquely determined to express investor's risk preference by risk aversion. However, mean-variance optimal portfolio's weights are extremely sensitive to the change in parameters, especially expected returns. In recent years, many researchers have shown interests in the approach of constructing portfolio, using risk due to the difficulty of estimating expected returns. Many studies attribute the better performance of these risk-based asset allocation approaches. In particular, risk parity portfolio attracts attention among practitioners and researchers. The approach equalizes risk contributions which is the decomposition of the total risk to each individual asset. The total risk can be the standard deviation of the portfolio return across all assets in general. In contrast, some studies propose the tail risk parity portfolio which equalizes risk contributions of downside risk measure (Alankar et al., 2012, Boudt et al., 2013). Few studies examine the effect of choosing risk measure and how to estimate downside risks. It is important to investigate the difference between general risk parity portfolio and tail risk parity portfolio.

In this paper, we construct tail risk parity portfolio using conditional value-at-risk(CVaR)[1] as downside risk. At first, we compare tail risk parity strategies with CVaRs estimated by three kinds of estimation methods (Delta-normal method, historical-simulation method, and Monte Carlo method), and examine the characteristics of the tail risk parity portfolios. Second, we implement the backtest for eighteen years using the historical data of Nikkei 225, Citi JPGBI (Japan government bond index), S&P500, and Citi USGBI, and we discuss the advantage of tail risk parity portfolio.

We find that the tail risk parity portfolio outperforms the usual risk parity portfolio. We decompose the difference of performance between them into three factors; 1. expected return, 2. distribution and 3. mutual dependence. The result shows that outperformance attributes to the expected return. Examining the distributions other than the normal distribution, the absolute return decreases, but the efficiency measure increases. The mutual dependence does not affect the difference of performance.

[1] CVaR is referred to as tail VaR, expected shortfall, conditional tail expectation.

2. Risk Parity Portfolio

We define each asset's risk contribution. The most commonly used definition is based on Euler's homogeneous function theorem. It is defined as follows,

$$RC_i = w_i \frac{\partial R(w)}{\partial w_i} = \frac{\partial R(w)}{\partial w_i / w_i} \quad (1)$$

where $R(w)$ is portfolio risk, and w_i is portfolio weight to asset *i*. Risk contribution is calculated as the sensitivity of the change in portfolio risk to the change in each weight. We satisfy the following equation.

$$R(w) = \sum_{i=1}^{n} RC_i \quad (2)$$

Equation (2) shows that the total portfolio risk equals the sum of each asset risk contribution by Euler's homogeneous function theorem.

Risk parity strategy utilizes standard deviation of portfolio return and equalizes its risk contribution across all assets. Using the method, the portfolio risk can be equally diversified to each asset.

Figure 1 shows portfolio weight and risk contribution for minimum variance portfolio, risk parity portfolio and equal weight portfolio, respectively. The bond weight and risk contribution of the minimum variance portfolio are largely composed. On the other hand, the equally-weighted portfolio holds completely well-balanced weights, but the risk contribution of stock becomes the largest portion of total risk. We have a 96% risk concentration on stock.

Maillard et al. (2009) show that the risk parity portfolio is located between the minimum variance portfolio and the equally-weighted portfolio. The risk parity portfolio is well-balanced between total risk minimization and portfolio diversification.

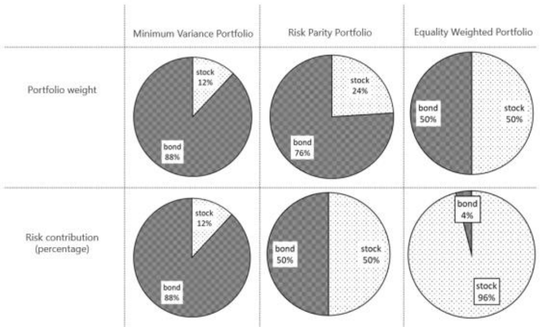

Figure 1: Portfolio weight and risk contribution[2]

3. Tail Risk Parity Portfolio

3.1 VaR (Value at Risk)

VaR represents the potential maximum loss on a given confidence level α.

$$VaR(\alpha) = \min_{1-\alpha}\{V : \mathrm{P}[-\mathrm{r_p} > V] \le (1-\alpha)\} \quad (3)$$

where r_P is a portfolio return.

Risk contribution of VaR can be defined as follows (Tasche (2000)),

$$RC_i^{VaR(\alpha)} = w_i \cdot E[-r_i \mid -r_P = VaR(\alpha)] \quad (4)$$

where r_i is a return of i-th asset. VaR is not coherent risk measure because it fails to satisfy the subadditivity. Yamai and Yoshiba (2002) argue that the risk contribution of VaR is highly sensitive to the portfolio's weight, and it is a serious problem for practical use.

3.2 CVaR (Conditional Value at Risk)

CVaR is defined as the average of the loss beyond the VaR, as follows.

$$CVaR(\alpha) = E[-r_P \mid -r_P \ge VaR(\alpha)] \quad (5)$$

Risk contribution of CVaR is defined as follows (Tasche (2000)),

$$RC_i^{CVaR(\alpha)} = w_i \cdot E[-r_i \mid -r_P \ge VaR(\alpha)] \quad (6)$$

The CVaR takes into account the maximum loss that is worse than the VaR and satisfies subadditivity. According to Yamai and Yoshiba (2002), in contrast to VaR, CVaR is insensitive to portfolio's weight. Therefore, we use CVaR as downside risk measure to construct tail risk parity portfolio.

3.3 Estimation of CVaR

We explain the following three methods of estimating CVaR. In this paper, we construct tail risk parity portfolio using those three kinds of estimation methods, and compare them.

3.3.1 Delta-Normal method

It is assumed that asset returns are normally distributed in the approach. As the portfolio return is a linear combination of normal variables, we can calculate the CVaR of the portfolio and risk contribution of asset *i* as follows

$$CVaR(\alpha) = -\sum_{i=1}^{n} w_i \mu_i + \frac{1}{1-\alpha} \sigma_P \phi[\Phi^{-1}(1-\alpha)] \quad (4)$$

$$RC_i^{CVaR(\alpha)} = -w_i \mu_i + \frac{w_i \sum_{j=1}^{n} \sigma_{ij} w_i}{\sigma_P} \frac{1}{1-\alpha} \phi[\Phi^{-1}(1-\alpha)] \quad (5)$$

where σ_{ij} is a covariance between returns of asset *i* and asset *j*. σ_P denotes standard deviation of the total portfolio returns. Φ^{-1} is a quantile function of the standard normal distribution

[2] We calculate the portfolio weight and risk contribution, using monthly data of S&P500 index and Citi USGBI index from January 1993 to July 2016.

and ϕ is a standard normal density function. This method is very practical and easy to use. However, many empirical studies show that returns of financial assets do not follow the normal distribution and the assumption of normally distributed financial returns underestimates VaR and CVaR.

3.3.2 Historical-Simulation method

This method is a non-parametric approach to estimate CVaR based on historical data. The CVaR (and VaR) can be calculated using the percentile of the empirical distribution corresponding to a given confidence level. This method can be applied to the non-normal distributions with heavy tails. However, the calculation is sensitive to the abnormal observation. This feature is inconvenient for constructing tail risk parity portfolio. Thus, we generate random numbers for the distribution which are obtained by kernel smoothing from the observed data. Suppose $\{x_1, x_2, \ldots, x_n\}$ denotes data observations for each asset. Kernel-smoothed cumulative distribution function (cdf) is

$$\hat{f}(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right), \tag{6}$$

where $K(\cdot)$ is a kernel function. It is the empirical distribution. Parameter h is the bandwidth or smoothing parameter. It controls the smoothness of the estimated cdf. We determine the bandwidth using the method of Matt and Jones (1994). We employ the Gaussian kernel function which is a commonly used.

$$K(u) = (2\pi)^{-1/2} e^{-u^2/2} \tag{7}$$

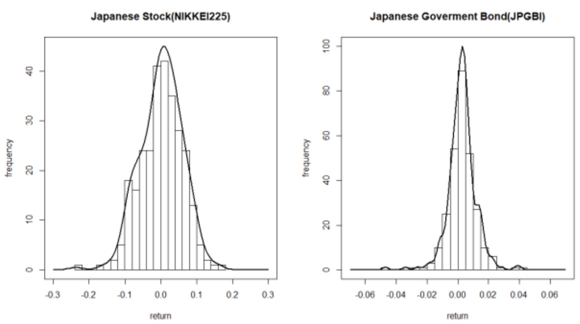

Figure 2: Kernel Smoothed functions

3.3.3 Monte Carlo method

The probability distribution and the dynamics of asset prices are simulated by generating random samples. It allows for any distribution (even non-normal distribution) and non-linear dependence. We generate random numbers which follows GH (Generalized Hyperbolic) distribution, and mutual dependence between assets represented by t-Copula. GH distribution is flexible enough to express fat tail and asymmetry. Copula describes dependence structure between each asset and can captures the tail of marginal distributions, unlike a linear correlation. We estimate GH distribution and t-Copula parameters by maximum likelihood method.

3.4 Formulation of asset allocation model with tail risk parity

We build tail risk parity portfolio which equalize all asset's risk contribution of CVaR. The model can be formulated as follows,

1 Sets
  F : set of foreign assets
2 Parameters
  N : number of assets
  d : risk-free interest rate of Japanese yen
  f : risk-free interest rate of U.S. dollar
  α : confidence level of CVaR
3 Decision variables
  w_i : portfolio weight of asset i

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=1}^{N}\left(\frac{RC_i^{CVaR(\alpha)}}{CVaR_P(\alpha)} - \frac{1}{N}\right)^2 \\ \text{subject to} \quad & \sum_{i=1}^{N} w_i + \sum_{j\in F}(f-d)w_j = 1 \\ & w_i \ge 0, i \in \{1,2,\ldots,N\} \end{aligned} \tag{8}$$

We solve the problem under the perfect hedging strategy for foreign assets. The hedging cost is the difference between U.S. and Japanese interest rate.

Problem (8) is difficult to solve using a commonly used mathematical programming tool because the objective function is non-convex, and RC and CVaR cannot be also expressed with the explicit function of the decision variables. Therefore, we used DFO[3] (Derivative Free Optimizer) method to solve the problem. However, the problem is dependent on an initial value, and then we set inverse volatility portfolio[4] as the initial portfolio weight.

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[3] The DFO method is the non-linear optimization method where the problems are solved without the derivative of the objective function. The type of the problem goes well with the DFO method. We used Numerical Optimizer/DFO added on the mathematical programming software package called Numerical Optimizer (ver. 18.1.0) developed by NTT DATA Mathematical System, Inc.

[4] The weights of inverse volatility portfolio are calculated as $w_i = \sigma_i / \sum_{j=1}^{N} \sigma_j$. This portfolio is equal to the risk parity portfolio when the correlations between assets are zero. Even if assets are correlated, it is expected that the portfolio takes a close value to tail risk parity or risk parity portfolio.

Table 1: Comparisons[5]

Case	1	2	3	4	5	6
Risk measure (estimation method of CVaR)	standard deviation	CVaR (Delta-Normal method)	CVaR (Historical-Simulation method)	CVaR (Monte Carlo method)	CVaR (Monte Carlo method)	CVaR (Historical-Simulation method)
Estimation of expected return	No	Yes	No ($\mu_i = 0$)	No ($\mu_i = 0$)	No ($\mu_i = 0$)	Yes
Probability distribution	normal	normal	historical	GH	normal	historical
Mutual dependence	linear correlation (=Gaussian copula)	linear correlation (=Gaussian copula)	Gaussian copula	Gaussian copula	t-copula	t-copula

4. Basic Analysis

We conduct the analysis for two-asset tail risk parity portfolio which consists of domestic stock and bond. We employ monthly data from January 1993 to July 2016 for Nikkei 225 stock and Citi JPGBI(Japan Government Bond Index). Summary statistics are shown in Table 2.

We set four kinds of the confidence level; 0.80, 0.85, 0.90, 0.95. The number of simulation paths is 20,000. We compare six cases in Table 1 in order to examine the difference between risk parity and tail risk parity portfolio.

Table 2: Statistics of return on an annual basis

	stock	bond
Mean	2.04%	3.24%
Standard deviation	20.20%	3.15%
Skewness	-0.323	-0.363
Exceed kurtosis	0.517	5.154

4.1 Expected return

We can construct risk parity portfolio without estimating expected returns. Some researchers say that this is one reason why risk parity portfolio has better performance than other portfolios. However, we need to estimate expected return to construct tail risk parity portfolio. Several studies have proved that it is difficult to estimate expected return. We pay attention to the fact that estimation errors of the expected return may affect the optimal portfolio. In our paper, we calculate average return in all period.

The difference of cases 1 and 2 in Table 1 is dependent on the expected return of asset because the CVaR is calculated in proportion to the standard deviation. Therefore, we compare the two cases, and examine the effect on the expected return for the tail risk parity portfolio.

Table 3: Comparison of the portfolio weights for the different expected returns

$\alpha = 0.80$	Case 1	Case 2
Stock	13.53%	12.19%
Bond	86.47%	87.81%

Table 3 shows the weights of each portfolio. Expected returns are 2.04% for stock and 3.24% for bond. The stock weight of tail risk parity portfolio is less than that of risk parity portfolio. The reason is that the asset with relatively higher expected return is allocated more in the tail risk parity portfolio. In addition, we find the difference tends to decrease as the confidence level becomes higher.

4.2 Distribution

We examine the effect on the distribution to compare case 1 and case 3(Historical-Simulation method) or case 4(Monte Carlo method) in Table 1.

Table 4: Comparison for the different distribution

$\alpha = 0.80$	Case 1	Case 3	Case 4
Stock	13.53%	12.59%	12.49%
Bond	86.47%	87.41%	87.51%
$\alpha = 0.95$	Case 1	Case 3	Case 4
Stock	13.53%	14.37%	13.75%
Bond	86.47%	85.63%	86.25%

Table 4 shows the weights of each portfolio in 0.80 and 0.95 confidence levels, respectively. Bond has lower skewness and higher kurtosis than stock.

According to cases 3 and 4, the weight of bond in tail risk parity portfolio decreases as the confidence level becomes higher. Examining the relationship between confidence level and distribution is our future task.

4.3 Mutual dependence

Describing the mutual dependence, non-linear correlation can be involved in the tail risk parity portfolio whereas

[5] $\mu_i = 0$ indicates asset return is normalized so that each mean return can be zero.

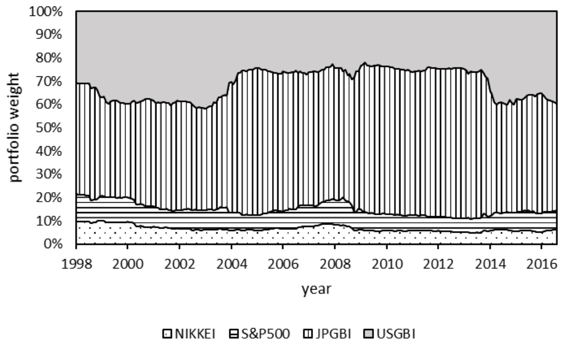

Figure 3: Weight of risk parity portfolio

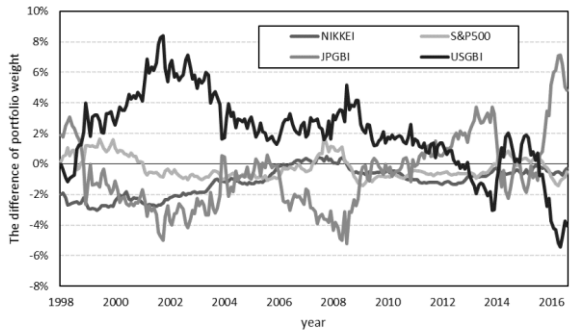

Figure 4: Expected return $(\alpha = 0.80)$

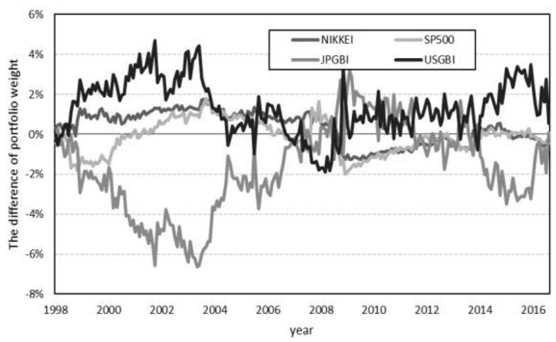

Figure 5: Historical distribution $(\alpha = 0.95)$

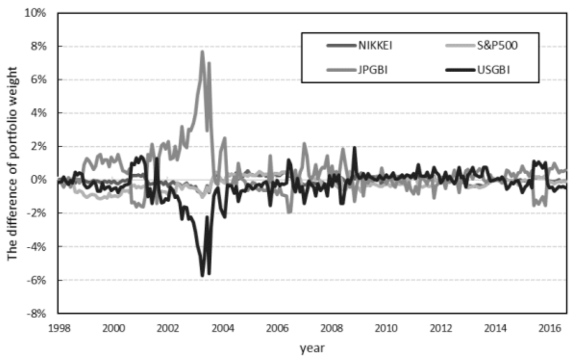

Figure 6: GH distribution $(\alpha = 0.80)$

Table 7: Backtest return on an annual basis

Case (See Table 1)	1	2	3	4	5	6
Mean	3.259%	3.296%	3.248%	3.205%	3.245%	3.272%
Standard deviation	2.683%	2.633%	2.657%	2.655%	2.677%	2.610%
Skewness	-0.604	-0.544	-0.577	-0.673	-0.601	-0.548
Exceed kurtosis	2.705	2.716	2.651	2.950	2.730	2.701
95%-CVaR	1.719%	1.681%	1.698%	1.716%	1.719%	1.660%
Maximum Drawdown	-6.593%	-6.123%	-6.356%	-6.504%	-6.590%	-6.096%
Sharpe ratio	1.134	1.170	1.141	1.126	1.132	1.171
CVaR ratio[6]	0.145	0.151	0.147	0.143	0.140	0.151

correlation coefficient is forced to be involved in the risk parity portfolio. We compare cases 1 and 5 in Table 1, and examine the effect on the mutual dependence.

Table 5: Comparisons for different mutual dependence

$\alpha = 0.80$	Case 1	Case 5
Stock	13.53%	13.53%
Bond	86.47%	86.47%

Table 5 shows the weights of each portfolio in 0.80 confidence level. We find the mutual dependence is not effective.

5. Backtest

It is well-known that the risk and dependence of financial assets are time-varying, which means that the optimal tail risk parity portfolio also change over time.

Suppose we invest four assets; Japanese stock and bond, U.S. stock and bond. The portfolio is rebalanced each first day of the month, and risk contribution of CVaR is estimated in a rolling window of sixty months. We implement the backtest in the following setting.

Data : Japanese Government Bond index – Citi JPGBI
Japanese Stock index – Nikkei 225

[6] CVaR ratio $= (\overline{r_P} - r_f)/CVaR(\alpha)$, where $\overline{r_P}$ is expected portfolio return and r_f is risk-free rate (1 month Japanese Yen LIBOR)

U.S. Government Bond index – Citi USGBI

U.S. Stock index–S&P500

Period: January 1993 – July 2016, monthly data

Currency hedging strategy: perfect hedging on a yen basis

Hedge cost: difference between U.S. and Japanese interest rate (one month LIBOR)

Number of simulation paths: 20,000 paths

We examine the results of the backtest as well as the basic analysis. We show the results for the confidence level where the difference of the portfolio weights between tail risk parity portfolio and risk parity portfolio is the largest (Figure 3 shows risk parity portfolio's weight).

5.1 Expected return

Figure 4 shows the difference of portfolio weights (tail risk parity portfolio minus risk parity portfolio) between cases 1 and 2 as in the basis analysis. The portfolio weight of U.S. bond has increased relatively toward 2002 due to the rise in expected return of U.S. bond.

As shown in Table 7, the tail risk parity portfolio outperforms the risk parity portfolio due to the effect of expected return.

5.2 Distribution

Similarly, Figures 5 and 6 show the difference of portfolio weights for different distributions, respectively.

Figure 5 shows the difference calculated using the historical simulation method between cases 1 and 3. The weight of Japanese bond has decreased relatively due to the decrease in skewness of the return. Figure 6 shows the difference calculated using Monte Carlo method under the GH distribution between cases 1 and 4. In contrast, the weight of Japanese bond has increased relatively due to the increase in kurtosis in 2003.

Table 7 indicates the absolute return goes down but the efficiency index goes up in the historical simulation. However, various statistics and efficiency index go down in Monte Carlo method under the GH distribution, compared with risk parity portfolio. The reason is that overinvesting Japanese Government Bonds has greatly influenced the 2003 VaR shock[7] in Japan.

5.3 Mutual dependence

The differences in the portfolio weights between cases 1 and 5 remain within the range of 1% in all period, and then we could not find the effect due to the non-linear dependence.

6. Conclusion

We compare the tail risk parity strategy using the following three estimation methods of CVaR; Delta-normal method, Historical-simulation method and Monte Carlo method. We also clarify the difference of risk parity portfolio and tail risk parity portfolio due to the following three factors; expected return, distribution and mutual dependence.

In the basic analysis, we find we invest in the assets with higher expected return and skewness in the tail risk parity portfolio. This result is reasonable to the expected utility theory. On the other hand, we tend to invest in the assets with higher kurtosis at low confidence level. This result is the opposite to the expected utility theory.

We also implement the backtest using historical data of Japanese stock and bond, U.S. stock and bond. The portfolio return of tail risk parity with historical-simulation method, has declined, but the efficiency index is rising. On the other hand, Monte Carlo method assuming GH distribution, various statistics and efficiency index deteriorated compared with usual risk parity portfolio. In this paper, we could not find the effect of non-linear dependence.

In the future research, we need to determine how to set parameters and how to decide the distribution to use the tail risk parity strategy in practice. We also need to compare with different risk parity strategies using downside risk measures.

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[7]The 10 year JGB yield triples from 0.5% in June 2003 to 1.6% in September 2003.