Optimal asset allocation with risk-adjusted implied return distribution based on the recovery theorem

Yuta Hibiki†
Graduate School of Science and Technology, Keio University
Yokohama, Japan
Email: y.hibiki@keio.jp

Takuya Kiriu
Mitsubishi UFJ Trust Investment Technology Institute Co., Ltd.
Tokyo, Japan
Graduate School of Science and Technology, Keio University
Yokohama, Japan
Email:kiriu@mtec-institute.co.jp

Norio Hibiki
Faculty of Science and Technology, Keio University
Yokohama, Japan
Email: hibiki@ae.keio.ac.jp

***Abstract*— In recent years, some previous papers show that better performances are obtained by solving asset allocation problem using risk-adjusted implied distribution estimated from market option prices, compared with historical distribution. (Kostakis et al. [8], Kimura and Hibiki [7]) However, Kiriu and Hibiki [5] show that it gives worse performance than implied risk-neutral distribution (RND) because it is risk-adjusted by a backward-looking approach. In this paper, we solve the optimal asset allocation problem with risk-adjusted implied return distribution based on the recovery theorem so that it can be also adjusted by a forward-looking approach. The recovery theorem is originally proposed by Ross [9], and it is generalized by Jensen et al. [4]. We examine the empirical performance for the two risk-adjusted methods, but we call them "RT" for simplicity hereafter. There are several previous studies of applying the RT to simple investment strategies. However, the performance in the optimal asset allocation problem has not been studied. We employ the RT in a finite state model, and examine how we discretize states. We estimate distributions with different ranges of states, and we find that the first moment is strongly related to the percentage of distributions included in the range. Therefore, we employ dynamic ranges based on the percentage of distributions. We solve the four asset allocation problem with two stocks and two bonds. We use historical distribution, implied RND, and risk-adjusted implied distributions by RT as distribution of stocks and compare the performance. We implement the backtest using dataset from 2005 to 2018. At first, we examine the performance for asset allocation with a risk-adjusted implied distribution of a single stock, and we find that it gives a better performance than historical distribution and implied RND. In addition, we utilize the risk-adjusted implied distribution of two stocks, and examine the performance. We find it is better than a single stock case.**

Keywords*— *financial engineering, optimal asset allocation, recovery theorem

I. INTRODUCTION

It is necessary to estimate the return distributions of assets to solve the optimal asset allocation problem. It is common to use a distribution estimated by historical asset price data. However, a weak point of using historical distribution is that it is difficult to cope with sudden changes of market environment due to being affected by past data. Then, in recent years, some previous papers show that better performances are obtained by solving asset allocation problem using risk-adjusted implied distribution estimated from option prices, compared with historical distribution. (Kostakis et al. [8], Kimura and Hibiki [7]). However, Kiriu and Hibiki [5] show that it gives a worse performance than implied risk-neutral distribution (RND) because it is risk-adjusted by a backward-looking approach.

Recently, Ross [9] developed the recovery theorem (RT) which enables implied RND to be risk-adjusted by the forward-looking approach using option data under some assumptions. In addition, Jensen et al. [4] proposed the generalized RT which relaxes the time-homogeneity of Ross's RT. We call these two theorems "RT" for simplicity hereafter. This gives the forward-looking risk-adjusted method without historical data for the implied RND, and it is expected to estimate the better distribution.

There are several previous studies of applying the RT to simple investment strategies. However, to the best of our knowledge, the strategies constructed using distribution adjusted by the RT has not been studied in the optimal asset allocation problem.

In this paper, we examine the effect of using the RT in the optimal asset allocation, through the comparison of the performance of the risk-adjusted distribution by the RT with those of historical distribution and implied RND. Specifically, we solve the asset allocation problem by the one-period model of minimizing conditional value at risk (CVaR), and implement the backtest using dataset from 2005 to 2018. We consider the problem assuming a Japanese investor who invests in the four assets; domestic stock, foreign stock, domestic bond, and foreign bond. These four assets are commonly used to determine asset allocation by Japanese investors. For example, the Government Pension Investment Fund of Japan defines the reference portfolio based on the four assets.

We use a historical distribution, implied RND, and risk-adjusted implied distributions by RT as distributions of stocks and compare the performance. We find the risk-adjusted implied distribution outperforms historical distribution and implied RND in terms of the downside risk and the efficiency of the portfolio. Moreover, the effect is even greater if risk-adjusted implied distributions used for two-stock case than for single-stock case.

II. RECOVERY THEOREM

The RT is the theorem which shows that the real world probability distribution and pricing kernel representing investor's risk preference can be recovered from risk-neutral distribution (state price) estimated from option prices under the following two assumptions.

1. The market is complete and there are no arbitrage opportunities
2. It is assumed that a representative investor follows time additive intertemporal expected utility theory

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The state of market s_i is represented by discrete state space $\{s_1, s_2, \ldots, s_S\}$ and we define each state as the return of option underlying asset, r_i. Let $\pi_{t,j}$ denote the state price for the transition from s_1 at the present time to s_j at time *t* and $q_{t,j}$ risk neutral probability, $f_{t,j}$ the real world transition probability, $m_{t,j}$ pricing kernel. In this paper, we estimate the real world probability distribution using the RT by the following three procedures according to the previous studies, such as Kiriu and Hibiki [6], and Ito et al. [3].

There are two types of the recovery theorem; time homogeneous case and non-homogeneous case. We call the former RRT (Ross's RT), and the latter GRT (Generalized RT) in order to distinguish them hereafter. In this paper, we employ them to obtain the risk-adjusted implied distribution.

A. Estimating the state price matrix $\boldsymbol{\Pi}$ from option data

We estimate the state price matrix $\boldsymbol{\Pi}$ using the relation between option price and state price introduced by Breeden and Litzenberger [2]. The state price $\pi(k,t)$ is expressed using call option price $c(k,t)$ as,

$$\pi(k,t) = \frac{\partial^2 c(k,t)}{\partial k^2} \quad (t = 1, \ldots, T) \qquad (1)$$

where *k* is a strike price and *t* is a maturity.

The state price in Equation (1) is continuously distributed. The RT by Ross [9] and Jensen et al. [4] is developed under a finite state space, and therefore we obtain the state price matrix $\boldsymbol{\Pi}$ by discretizing $\pi(k,t)$ numerically.

B. Estimating the normalized marginal utility and subjective discount factor

At first, we explain the GRT. We estimate the normalized marginal utility vector $\boldsymbol{h}$ and subjective discount factor δ from the state price matrix $\boldsymbol{\Pi}$ for GRT. A utility maximization problem of a representative investor with a time additive intertemporal utility is formulated as

$$\max_{c_1} \quad U(c_1) + \delta^{\tau_t} \sum_{j=1}^{S} f_{t,j} U(c_j) \quad (t = 1, \ldots, T), \qquad (2)$$

$$\text{subject to} \quad c_1 + \sum_{j=1}^{S} \pi_{t,j} c_j = \omega \quad (t = 1, \ldots, T), \qquad (3)$$

where c_j is consumption at state s_j, S is the number of states, $U(c_j)$ is a utility of a representative investor with c_j, τ_t is the length of time to *t*, and ω is an amount of wealth of a representative investor. The optimization problem can be solved employing the method of Lagrange multiplier, and we can get the following equation.

$$\delta^{\tau_t} f_{t,j} U'(c_j) = U'(c_1)\pi_{t,j} \quad (t = 1, \ldots, T; j = 1, \ldots, S) \qquad (4)$$

We define a normalized marginal utility for state s_j as $h_j = U'(c_j)/U'(c_1)$ so that the marginal utility $U'(c_1)$ can be equal to one for consumption c_1. Equation (4) can be rewritten as

$$\pi_{t,j} = \delta^{\tau_t} f_{t,j} h_j \quad (t = 1, \ldots, T; j = 1, \ldots, S). \qquad (5)$$

It can be rewritten using matrices as

$$\mathbf{\Pi} = \mathbf{DFH} \Leftrightarrow \mathbf{F} = \mathbf{D}^{-1}\mathbf{\Pi}\mathbf{H}^{-1} \qquad (6)$$

where $\mathbf{H}$ is $diag(1, h_2, \ldots, h_S)$, $\mathbf{D}$ is $diag(\delta^{\tau_1}, \delta^{\tau_2}, \ldots, \delta^{\tau_T})$. In addition, since $\mathbf{F}$ is a stochastic matrix,

$$(\mathbf{D}^{-1}\mathbf{\Pi}\mathbf{H}^{-1})\mathbf{e} = \mathbf{e} \Leftrightarrow \mathbf{\Pi}\mathbf{H}^{-1}\mathbf{e} = \mathbf{De} \qquad (7)$$

or

$$\begin{bmatrix} \pi_{1,1} & \cdots & \pi_{1,S} \\ \vdots & \ddots & \vdots \\ \pi_{T,1} & \cdots & \pi_{T,S} \end{bmatrix} \begin{bmatrix} 1 \\ h_2^{-1} \\ \vdots \\ h_{S-1}^{-1} \\ h_S^{-1} \end{bmatrix} = \begin{bmatrix} \delta^{\tau_1} \\ \vdots \\ \delta^{\tau_T} \end{bmatrix}. \qquad (8)$$

Equation (8) is not easy to solve due to nonlinearity. We calculate the first-order of Taylor series of δ^{τ_t} in order to solve it linearly. The Taylor series of δ^{τ_t} at $\delta = \delta_0$ is $\delta^{\tau_t} = a_t + b_t\delta, \ \ a_t = -(t-1)\delta_0^{\tau_t}, \ \ b_t = t\delta_0^{\tau_t - 1}$, for the linear approximation. Therefore Equation (8) is approximated as

$$\begin{bmatrix} \pi_{1,1} & \cdots & \pi_{1,S} \\ \vdots & \ddots & \vdots \\ \pi_{T,1} & \cdots & \pi_{T,S} \end{bmatrix} \begin{bmatrix} 1 \\ h_2^{-1} \\ \vdots \\ h_{S-1}^{-1} \\ h_S^{-1} \end{bmatrix} = \begin{bmatrix} a_1 + b_1\delta \\ \vdots \\ a_T + b_T\delta \end{bmatrix}. \qquad (9)$$

Moreover, it is rewritten as

$$\begin{bmatrix} -b_1 & \pi_{1,2} & \cdots & \pi_{1,S} \\ \vdots & \vdots & \ddots & \vdots \\ -b_T & \pi_{T,2} & \cdots & \pi_{T,S} \end{bmatrix} \begin{bmatrix} \delta \\ h_2^{-1} \\ \vdots \\ h_S^{-1} \end{bmatrix} = \begin{bmatrix} a_1 - \pi_{1,1} \\ \vdots \\ a_T - \pi_{1,T} \end{bmatrix}. \qquad (10)$$

Equation (10) is rewritten as $\mathbf{B}\boldsymbol{h}_\delta = \boldsymbol{a}_\pi$ using a matrix $\mathbf{B}$ and vectors $\boldsymbol{h}_\delta$, and $\boldsymbol{a}_\pi$. The parameters $\boldsymbol{h}$ and δ are estimated by solving the following optimization problem.

$$\min_{\boldsymbol{h}_\delta} \|\mathbf{B}\boldsymbol{h}_\delta - \boldsymbol{a}_\pi\|_2^2 \qquad (11)$$

$$0 < \delta \leq 1 \qquad (12)$$

$$h_j^{-1} > 0 \quad (j = 1, \ldots, S) \qquad (13)$$

Up to here, we explain the GRT, hereafter, we explain the RRT. We estimate the transition state price matrix $\mathbf{P}$ for RRT. Denote the matrix whose last row is removed from $\mathbf{\Pi}$ by $\mathbf{\Pi}_{-T}$ and the matrix whose first row is removed from $\mathbf{\Pi}$ by $\mathbf{\Pi}_{-1}$. In addition, denote the first row of $\mathbf{\Pi}$ by $\boldsymbol{\pi}_1$, the first row of $\mathbf{P}$ by $\boldsymbol{p}_1$. Since both $\boldsymbol{\pi}_1$ and $\boldsymbol{p}_1$ represent a state price distribution at current state, $\boldsymbol{\pi}_1 = \boldsymbol{p}_1$. Using these, we can compute $\mathbf{P}$ as follows.

$$\min_{\boldsymbol{P}} \|\mathbf{\Pi}_{-T}\mathbf{P} - \mathbf{\Pi}_{-1}\|_2^2 \qquad (14)$$

$$\text{subject to} \quad \boldsymbol{\pi}_1 = \boldsymbol{p}_1 \qquad (15)$$

$$p_{i,j} \geq 0 \quad (i,j = 1, \ldots, S) \qquad (16)$$

where $p_{i,j}$ is the transition state price from s_i to s_j.

Some previous papers (Audrino et al. [1], Kiriu and Hibiki [6], Ito et al. [3]) show that these problems ((11) ~ (13), (14) ~ (16)) are ill-posed, and the solution is sensitive to the noise included in the option price data. Then, we employ the method of Kiriu and Hibiki [6] for RRT, Ito et al. [3] for GRT which stabilizes the solution by introducing the regularization term in consideration of prior information in the objective function in this paper. For GRT, the objective function considering prior information is as follows.

$$\min_{\boldsymbol{h}_\delta} \|\mathbf{B}\boldsymbol{h}_\delta - \boldsymbol{a}_\pi\|_2^2 + \varsigma \sum_{s=2}^{S} \left(h_s^{-1} - \overline{h_s^{-1}}\right)^2 \tag{17}$$

where ς is the regularization parameter, $\overline{h_s^{-1}}$ is the prior information of the marginal utility. In this paper, we set two types of the prior information, (i) risk neutral utility (ii) CRRA utility function. If we want to employ risk neutral utility, we should set $\overline{h_s^{-1}} = 1 (s = 2, \ldots, S)$. If we want to employ CRRA utility function, since we can express reciprocal of the marginal utility, h_s^{-1} as $(1 + r_s)^{\gamma_R}$ using relative risk aversion, γ_R, we estimate γ_R by solving (11) ~ (13) , and we set $\overline{h_s^{-1}}$.

C. Estimating the real world transition probability matrix F

The real world transition probability $f_{t,j}$ can be calculated by (18) using the state price matrix $\mathbf{\Pi}$, marginal utility vector $\boldsymbol{h}$, and subjective discount factor δ for GRT.

$$f_{t,j} = \frac{\pi_{t,j}}{\delta^{\tau_t} h_j} \quad (t = 1, \ldots, T; j = 1, \ldots, S) \tag{18}$$

In the case of RRT, The real world transition probability from s_i to s_j, $f_{i,j}$ can be calculated by (19).

$$f_{i,j} = \frac{1}{\lambda} \frac{v_j}{v_i} p_{i,j} \; (i, j = 1, \ldots, S) \tag{19}$$

where λ is the maximum eigenvalue of $\mathbf{P}$, and $\boldsymbol{v}$ is an eigenvector associated with λ.

The real world transition probability $f_{i,j}$ is derived discretely. However, we transform the discrete function to the continuous probability function $f(r)$ by spline interpolation, where *r* is a return.

III. Estimating Historical Distribution

We employ historical or implied distribution for stocks, and historical distribution for bonds to solve the four asset allocation problem.

We use the kernel density estimation to estimate the historical distribution in a non-parametric way because the implied distribution is non-parametrically derived.

Let $\{x_1, x_2, \ldots, x_n\}$ be n observation data of historical return. The probability distribution function $\widehat{f_b}(x)$ estimated by kernel density estimation is given as

$$\widehat{f_b}(x) = \frac{1}{nb} \sum_{i=1}^{n} K\left(\frac{x - x_i}{b}\right) \tag{20}$$

where $K(\cdot)$ is a kernel function, b is the bandwidth of the kernel which affects the smoothness of probability distribution function. In this paper, we employ Gaussian kernel represented by (21) which is often used as kernel function $K(\cdot)$.

$$K(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{21}$$

Selection of the bandwidth strongly affects the results. If the bandwidth is too large, the probability density function is oversmoothed, whereas it is undersmoothed if the bandwidth is too small. In this paper, we employ the bandwidth computed by the commonly used Silverman's rule of thumb as follows.

$$b = \frac{0.9A}{n^{1/5}} \tag{22}$$

Let σ_x denote the standard deviation of $\{x_1, x_2, \ldots, x_n\}$, and QR_x denote the interquartile range of $\{x_1, x_2, \ldots, x_n\}$. A is denoted by $\min\{\sigma_x, QR_x/1.34\}$.

IV. Optimal Asset Allocation Model

We compute the optimal asset allocation of minimizing the conditional value at risk for an 80% confidence level. We adopt the perfect hedging strategy for foreign assets. The hedging cost is calculated as the difference of risk-free interest rates between Japanese yen and U.S. dollar. In addition, we assume that short sales are not allowed. The notations used in the model are:

1. Sets
   - i : superscript representing a path
   - j : subscript representing an asset
   - F : set of foreign assets
2. Parameters
   - I : number of sample paths
   - n : number of assets
   - d : risk-free interest rate of Japanese yen
   - f : risk-free interest rate of U.S. dollar
   - β : confidence level of CVaR (ex. $\beta = 0.80$)
   - r_j^i : rate of return of asset j on path i
3. Decision variables
   - x_j : portfolio weight of asset j
   - α_β : β-VaR (Value at Risk)
   - q^i : shortfall below VaR on path *i*

The model can be formulated as follows.

$$\begin{aligned}
&\text{Minimize} && \text{CVaR} = \alpha_\beta + \frac{1}{(1-\beta)I} \sum_{i=1}^{I} q^i \\
&\text{Subject to} && \sum_{j=1}^{n} x_j + \sum_{j \in F} (f - d) x_j = 1 \\
& && \sum_{j=1}^{n} r_j^i x_j + \alpha_\beta + q^i \geq 0 \quad (i = 1, \ldots, I) \\
& && q^i \geq 0 \quad (i = 1, \ldots, I) \\
& && x_j \geq 0 \quad (j = 1, \ldots, n)
\end{aligned} \tag{23}$$

V. Preliminary Analysis

We need to select the number of states and the range of states because of a finite state space model. However, we have the trade-off relationship between the number of states and the range of states. The increase in the number of states makes the ill-posed problem difficult to solve stably, whereas the expansion of the range with fixed number of states makes it difficult to compute the expected return precisely which is affected by the wide interval between states in the middle part of the distribution. Suppose we fix the number of states for computation. The selection of the range of states affects the interval between states and the tail of return distribution. If we adopt the constant state range to fix the interval, the tail may be cut largely, and it also affects the optimal asset allocation. Then, we set the time-dependent state range dynamically.

Specifically, we determine that the range of states is set dynamically so that it includes the most part of the implied RND obtained from the option data for the longest maturity. It is expected that the range can also cover the one-period (one month) distribution used in the asset allocation problem. Next, we discuss how much percentage (probability) of the implied RND is included in the range.

Fig. 1 shows the relation between the percentage of the risk-neutral distribution of foreign stock for the longest maturity in the range and the expected return (the first moment of distribution) of the distribution estimated using GRT. There are three curves in Fig. 1, each of which represents the relation on the day when the values of VIX (volatility index) are maximum, minimum, and median, respectively. For example, suppose that the percentage of implied RND for the longest maturity is set to 90% on March 16, 2016. We obtain 0.15% expected return. Note that the level of the expected return of the distribution estimated using GRT on November 20, 2008 is expressed on the right vertical axis in Fig. 1 because of the different level from those of the other two days.

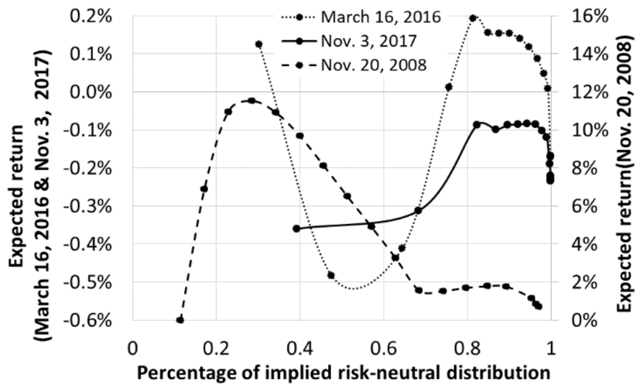

Fig. 1. Relationship between the percentage of RND and the expected return estimated using GRT.

According to Fig. 1, the expected return is dependent on the percentage of the distribution included in the range of states. We find the expected returns calculated are similar to those of original distribution for the wide range, and they are stable for the percentage of 85 to 95% on any day. Therefore, we change the range of states dynamically so that a 95% of the distribution can be included in the range.

VI. Backtest

A. Setting

In this paper, we examine whether risk-adjusted implied distributions using RRT and GRT in the forward-looking approach make better performance than both historical distribution and implied RND. Then, we compare the performances among the five return distributions in Table 1. In addition, we examine three combinations of the implied distribution of assets each of which consists of domestic stock, foreign stock, and domestic and foreign stocks, respectively. The dataset is as follows.

Asset price index
1) Domestic stock: Nikkei 225 stock index
2) Domestic bond: FTSE JPGBI (Japanese Government Bond index)
3) Foreign stock: S&P 500 stock index
4) Foreign bond: FTSE USGBI (U.S. Government Bond index):

Risk-free interest rate
1) Japanese yen: one month JPY LIBOR
2) U.S. dollar: one month USD LIBOR

Option price
1) Japanese stock option: Nikkei 225 index option
2) U.S. stock option: S&P 500 index option

We use the data of ATM (At The Money) option and OTM (Out of The Money) option in terms of the liquidity.

We utilize the historical distributions for domestic and foreign bonds, which are estimated by kernel density estimation using rolling window size of 60 months for all cases. The dependency structure between assets is expressed by t-copula for representing the tail dependency. Then, we generate 20,000 sample paths for the joint probability distributions of the four assets' returns, and compute the optimal asset allocation. The portfolio is rebalanced at the beginning of each month. We implement the backtest using dataset from January 2005 to September 2018, and we set a 95% of implied RND as the range of states according to the preliminary analysis. The number of states is 51.

B. Performance in the case of a single stock

- *Domestic stock case*

We show the summary statistics of performance of the model with each distribution of domestic stock in Table 2. All the values are on a monthly basis. The best values are bold among five cases, and the worst values are italic. Mean returns are almost the same as each other. However, we find the "RRT" (Case 3) gives the best performance in terms of standard deviation, skewness, 80%-CVaR, Maximum Drawdown (MDD), Sharpe ratio and CVaR ratio. The performances of the risk-adjusted distributions by the RT (Case 3 to 5) are also superior to those of historical distribution (Case 1) and implied RND (Case 2). Therefore, we can improve the performance by employing the risk-adjusted implied distribution in the forward looking approach.

- *Foreign stock case*

Table 3 shows the summary statistics of performance with each distribution of foreign stock. The "RRT" (Case 3) has the best values of mean return, three risk measures (standard deviation, 80%-CVaR, MDD), and two performance measures (Sharpe ratio, CVaR ratio). We can also improve the performance by employing the risk-adjusted implied distribution even for foreign stock.

The relative results among five cases are totally similar to Table 2, but the performance of "RRT" of foreign stock case is superior to that of domestic stock case. We show the difference of weights of S&P 500 index in the optimal portfolio in order to explain the reason in Fig. 2.

We find the S&P500 weight decreases during the financial crisis from 2007 to 2008, and increases during the period of recent strong economy in the U.S. This gives the highest mean return and performance measures.

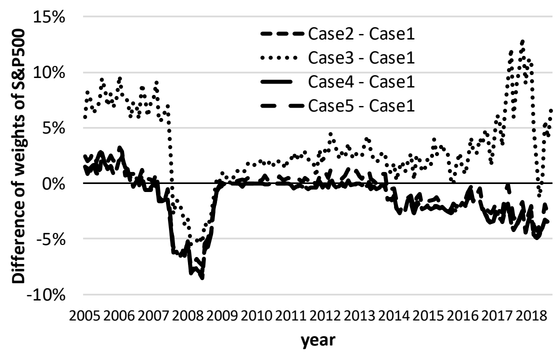

Fig. 2. Difference of weight of each implied distribution of S&P 500 (Case 2 to 5) to the weight of historical distribution (Case 1)

C. Performance in the case of two-stocks

Table 4 shows the summary statistics for two-stock case. The "RRT" (Case 3) has the best values of mean return, three risk measures and two performance measures as well as the foreign stock case. Moreover, these values are much better than the domestic stock case, and slightly better than the foreign stock case. The relative results among five cases are totally similar to Table 2, but the performance of "RRT" of foreign stock case is superior to that of domestic stock case. We find the use of risk-adjusted implied distribution of two stocks makes the performance improve further than a single stock case.

TABLE 1. Comparison of distributions

Case & Name	1. Historical	2. RND	3. RRT	4. GRT-N	5. GRT-CRRA
Distribution	historical	implied RND	risk-adjusted	risk-adjusted	risk-adjusted
Prior information	–	–	risk-neutral utility	risk-neutral utility	CRRA utility

TABLE 2. Backtest return on a monthly basis in domestic stock case

Case & Name	1. Historical	2. RND	3. RRT	4. GRT-N	5. GRT-CRRA
Mean	*0.186%*	0.194%	0.190%	0.193%	**0.195%**
Standard deviation	0.603%	*0.618%*	**0.583%**	0.615%	0.606%
Skewness	0.195	*0.192*	**0.212**	*0.192*	0.193
Exceed kurtosis	-0.004	0.180	-0.016	**0.185**	0.173
80%-CVaR	0.643%	*0.656%*	**0.612%**	0.651%	0.635%
Maximum Drawdown	3.78%	*4.18%*	**3.46%**	4.17%	4.07%
Sharpe ratio	*0.282*	0.288	**0.299**	0.288	0.296
CVaR ratio[1]	*0.265*	0.271	**0.285**	0.272	0.282

TABLE 3. Backtest return on a monthly basis in foreign stock case

Case & Name	1. Historical	2. RND	3. RRT	4. GRT-N	5. GRT-CRRA
Mean	*0.186%*	0.194%	**0.220%**	0.193%	0.197%
Standard deviation	0.603%	*0.611%*	**0.585%**	0.609%	0.602%
Skewness	**0.195**	0.175	0.180	*0.156*	0.167
Exceed kurtosis	*-0.004*	**0.197**	0.032	0.176	0.170
80%-CVaR	0.643%	*0.646%*	**0.596%**	0.646%	0.634%
Maximum Drawdown	3.78%	*4.18%*	**3.44%**	4.17%	4.06%
Sharpe ratio	*0.282*	0.293	**0.349**	0.292	0.302
CVaR ratio	*0.265*	0.277	**0.343**	0.275	0.286

TABLE 4. Backtest return on a monthly basis in two-stock case (domestic and foreign stocks)

Case & Name	1. Historical	2. RND	3. RRT	4. GRT-N	5. GRT-CRRA
Mean	*0.186%*	0.199%	**0.222%**	0.198%	0.205%
Standard deviation	0.603%	0.629%	**0.572%**	*0.630%*	0.614%
Skewness	**0.195**	0.127	0.158	*0.119*	0.143
Exceed kurtosis	*-0.004*	0.327	0.028	**0.340**	0.323
80%-CVaR	0.643%	0.671%	**0.572%**	*0.673%*	0.640%
Maximum Drawdown	3.78%	4.51%	**3.15%**	*4.57%*	4.32%
Sharpe ratio	*0.282*	0.292	**0.361**	0.289	0.309
CVaR ratio	*0.265*	0.273	**0.361**	0.270	0.296

[1] CVaR ratio $= \left(\overline{r_P} - r_f\right)/CVaR(\beta)$, where $\overline{r_P}$ is expected portfolio return and r_f *is* risk-free rate (one month Japanese yen LIBOR).

VII. Conclusion

In this paper, we examine the effect of using the RT in the optimal asset allocation problem by the one-period model of minimizing CVaR, through the comparison of the performance of the risk-adjusted distribution by the RT with those of historical distribution and implied RND.

We implement the backtest using dataset from 2005 to 2018, and obtain two findings. At first, the risk-adjusted implied distribution in the forward looking approach outperforms the historical distribution and implied RND in terms of the downside risk and the efficiency of the portfolio. Second, the performance of "RRT" gives the best performance for a single stock and two-stock cases among five cases of distributions. We also find the performance of "RRT" of foreign stock case is superior to that of domestic stock case, and the use of risk-adjusted implied distribution for two stocks makes the performance improve further than a single stock.

In the future research, we need to illustrate how the features of the RRT affect the asset allocation, and the reason why the performance can be improved by the RRT for practical use.

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