Rule of 126: How many years does it take for a value of installment savings account to double?

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1. Introduction

In finance, the rule of 72 is well known as the simple method for estimating an investment's doubling time[1]. This is a rule to find a combination of the number of years and the annual interest rate (rate of return) that will double the principal under a one-time deposit (a lump-sum investing). Specifically, the rule is that "Number of years × Annual interest rate = 72"(the interest rate is expressed as a percentage). For example, if the annual interest rate is 3%, it would take 24 years to approximately double ($1.03^{24} = 2.033$).

The principal doubles in the simple interest world by "Number of years × Annual interest rate = 100" and (rule of) 72 is smaller than 100, which would represent the compounding effect. This will help you feel more familiar with the compounding effect of investment and asset building because we find easily that "At what rate of interest and for how many years would it take to double the amount?". Rule of 72 is known as a very straightforward rule, but it is a rule for principal that can be deposited or invested in a lump sum. On the other hand, to the best of our knowledge, there is no such rule for installment saving deposits.

In this paper, we propose the "rule of 126" as a rule for installment savings or investments corresponding to the rule of 72 for a one-time deposit or a lump-sum investment. This is the rule that the principal can be doubled under the assumption of installment savings, which is described as "Number of years × Annual interest rate = 126". However, you need to understand this rule does not take into account the risk of making an investment as well as the rule of 72. [1]

The rule says that the future value approximately doubles the accumulated principal (the accumulated principal is approximately half the future value) if you accumulate for 42 years under 3% interest rate ($42 \times 3 = 126$), for example. Assuming you work for 42 years from age 23 to 65, it is easy to understand that you would only need half the amount of savings approximately if you can deposit or invest under the assumption of 3% interest rate (rate of return) as soon as you start working. If you want to save 20 million yen by accumulating the money over 504 months (42 years), monthly savings amount would be 19,793 yen under the 3% interest rate. On the other hand, using the 126 rule, the monthly savings amount is 19,841 yen ($= \frac{10 \text{ million yen}}{504 \text{ months}}$) since the total amount of principal to be saved is 10 million yen. We can almost approximate the amount by the rule of 126. The principal doubles in the simple interest world by "Number of years × annual interest rate = 200" and it also represents the compounding effect. In addition, we indicate the rules corresponding to other ratios, not doubling.

[1] When risk is taken into account, the geometric mean of the rate of return is different from the arithmetic mean. In addition, in the case of installment investment (dollar-cost averaging investment), the rate of return should be calculated taking into account the amount invested because the amounts invested increase gradually over time. Therefore, these points should be noted but if they are understood well, it can be fully useful. However, we explain the rule using risk-free installment savings deposit to avoid misunderstandings.

2. Rule of 72 for an initial one-time deposit

The rule of 72 is a simple and useful formula that is popularly used to calculate the number of years required to double an initial one-time deposit at a given interest rate or to calculate the interest rate to double at a given number of years. The formula is $nr = 0.72$, where n is the number of years and r is the interest rate. It can be rewritten by $n = 0.72/r$ or $r = 0.72/n$. Any combination of the number of years n and the interest rate r that satisfies $(1+r)^n = 2$ doubles the principal under compound interest. Substituting $r = 0.72/n$ into this equation, the value of $(1 + 0.72/n)^n$ is the ratio to the principal. The relationship between the number of years n and its ratio can be shown as in Figure1. In the case of $n = 9.1756$ ($r = 0.0785$), the value after n years is exactly twice the principal. Even in the case of the other year n, we can confirm that the future value doubles the principal approximately.

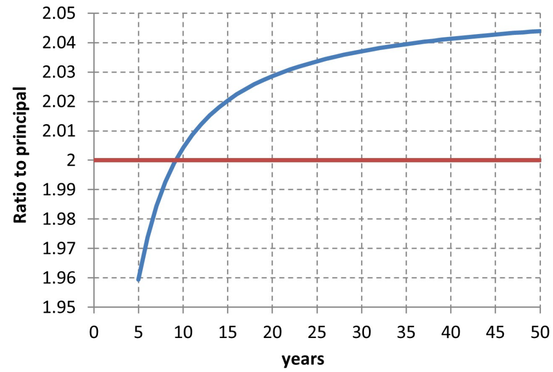

Figure 1: Rule of 72

If the future value of the initial one-time deposit is calculated under continuous compounding, the value a which is used for the rule can be calculated as follows, where y is the ratio to principal. The number used for the rule is 100 times the value of a because the interest rate is described at 100 times such that 3% is 3 for example.

$$y = e^{nr} \tag{2.1}$$

$$a = nr = \ln(y) \tag{2.2}$$

Therefore, the rule of doubling the principal is $a = \ln(2) = 0.693$, substituting $y = 2$, which yields the rule of 69.

3. Rule for installment savings deposit

An installment savings deposit generally accumulates a fixed amount of money each month. Suppose that the value M is deposited over mn times at the beginning of the period and FV is obtained at maturity as shown in Figure2, where n is the number of years, m is the number of times of accumulation in a year. The accumulation is made monthly for $m = 12$, quarterly for $m = 3$, and annually for $m = 1$. Denote r be an interest rate (annual rate). Since the future value FV is the sum of the future values of the amount M each period, the future value FV can be calculated using the amount M each period as follows.

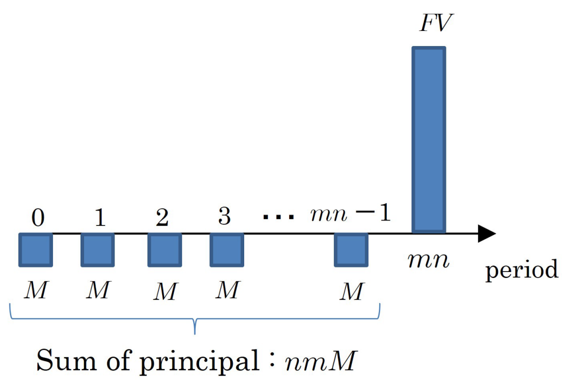

Figure 2: Cashflow stream of installment savings

$$FV = \underbrace{\left(1+\frac{r}{m}\right)^{mn} M}_{\text{FV of } M \text{ at time } 0} + \underbrace{\left(1+\frac{r}{m}\right)^{mn-1} M}_{\text{FV of } M \text{ at time } 1} + \cdots + \underbrace{\left(1+\frac{r}{m}\right)^{2} M}_{\text{FV of } M \text{ at time } nm-2} + \underbrace{\left(1+\frac{r}{m}\right) M}_{\text{FV of } M \text{ at time } nm-1}$$

$$= \left\{\sum_{t=1}^{mn}\left(1+\frac{r}{m}\right)^{t}\right\} M = \left[\frac{\left\{\left(1+\frac{r}{m}\right)^{mn}-1\right\}\left(1+\frac{r}{m}\right)}{\frac{r}{m}}\right] M \tag{3.1}$$

The sum of accumulated deposit is Mmn, which is the total principal. The ratio of future value to principal, y, is calculated in the followings.

$$y = \frac{FV}{Mmn} = \frac{\left\{\left(1+\frac{r}{m}\right)^{mn} - 1\right\}\left(1+\frac{r}{m}\right)}{nr} = \frac{\left\{\left(1+\frac{a}{mn}\right)^{mn} - 1\right\}\left(1+\frac{a}{mn}\right)}{a} \tag{3.2}$$

where $a = nr$. Let the equation make m approach infinity. The ratio can be calculated as follows. [2].

$$y = \lim_{m\to\infty} \frac{1}{a}\left\{\left(1+\frac{a}{mn}\right)^{mn} - 1\right\}\left(1+\frac{a}{mn}\right) = \frac{1}{a}\left(e^{a}-1\right) \tag{3.4}$$

The ratio y is a function of a. Independently of the combination of n and r, the ratio y is determined for the product a. Also, a is uniquely determined for y because this function is a strictly increasing function, Numerical calculation for $y = 2$ yields $a = 1.2564$.

However, a depends on n for other m. The values of $a(y; m, n)$ which are obtained by solving Equation (3.4) for the combination of m, n, and y are shown in Table1.

Table 1: The values of $a(y; m, n)$ for some combinations of m, n, and y

y	$m^{\dagger}$	$n=10$	$n=20$	$n=30$	$n=40$	$n=50$	y	$m \to \infty$
	1	1.9348	1.9202	1.9150	1.9123	1.9106		
3	4	1.9123	1.9081	1.9067	1.9060	1.9055	3	1.9038
	12	1.9067	1.9053	1.9048	1.9045	1.9044		
	1	1.2304	1.2436	1.2479	1.2500	1.2513		
2	4	1.2500	1.2532	1.2543	1.2548	1.2552	2	1.2564
	12	1.2543	1.2554	1.2557	1.2559	1.2560		
	1	0.7257	0.7438	0.7500	0.7531	0.7550		
1.5	4	0.7531	0.7579	0.7595	0.7603	0.7608	1.5	0.7627
	12	0.7595	0.7611	0.7616	0.7619	0.7620		

† $m = 1$: annually, $m = 4$: quarterly, $m = 12$: monthly

The values of $a(y; m, n)$ in Table 1 are different depending on the combinations of m and n, but are similar for a given y value. In particular, the values for $m = 12$ are almost the same as those for $m \to \infty$, regardless of n.

The ratios for two as that take the values close to $y = 2$ with $m = 12$ are shown in Figure 3. The horizontal axis on the left figure is the number of years (n), the horizontal axis on the right fighre is the interest rate (r). The vertical axis is the ratio of the future value to the accumulated principal (y). We find that a cannot be uniquely determined strictly in Figure 3 because the ratio (y) varies with the number of years (n) and the interest rate (r). However, we can also see that the approximate values are almost the same.

Next, let us find the integer that is closest to the value of $100a$ for $m = 12$ which is commonly applied in installment savings deposits, in order to determine the rules for the three different ratios, We find easily that it is better to choose $a = 0.76$ for $y = 1.5$. On the other hand, either $a = 1.25$ or $a = 1.26$ should be chosen for $y = 2$, and either $a = 1.90$ or $a = 1.91$ for $y = 3$. Therefore, we consider the minimum period of the installment years n_1 to be 10 years and the maximum years n_2 to be 40 and 50 years, and calculate the mean squared errors with respect to the ratio y for each period. Similarly, the mean squared errors are also calculated when the maximum value of the interest rate is set to 5 % and 10 %. We show the results in Table 2.

[2]Suppose the continuous compounding and continuous accumulation of deposit. The ratio can be calculated as follows.

$$y = \frac{1}{mn}\int_0^{mn} e^{(r/m)t}dt = \frac{1}{mn}\cdot\frac{1}{r/m}\left[e^{(r/m)t}\right]_0^{mn} = \frac{1}{nr}\left(e^{nr}-1\right) = \frac{1}{a}\left(e^{a}-1\right) \tag{3.3}$$

The same formula can be derived as Equation (3.2).

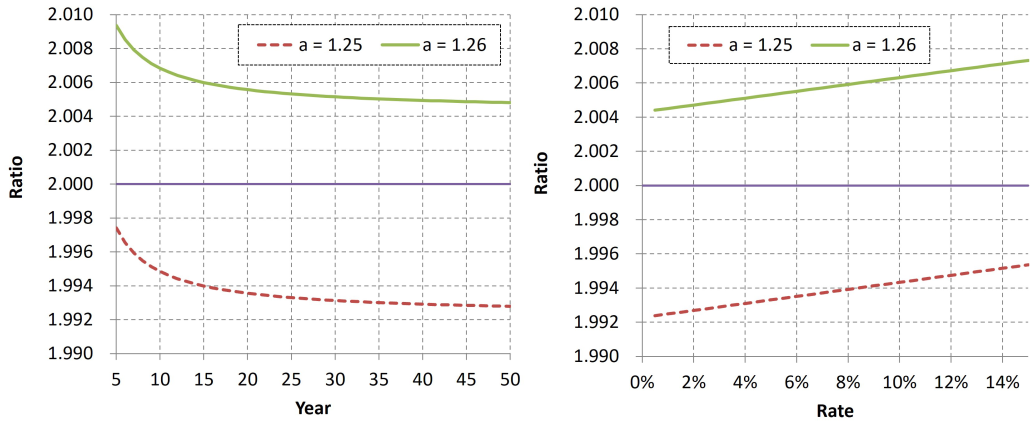

Figure 3: Comparison of the ratios for two as that take the values close to $y = 2$

Table 2: Mean squared errors

Years		$y = 2$		$y = 3$	
n_1	n_2	$a = 1.25$	$a = 1.26$	$a = 1.90$	$a = 1.91$
10	50	1.08%	$\underline{0.86\%}$	$\underline{0.69\%}$	0.70%
10	40	0.91%	$\underline{0.77\%}$	0.62%	$\underline{0.58\%}$
Rate		$a = 125$	$a = 126$	$a = 190$	$a = 191$
∼5%		0.57%	$\underline{0.38\%}$	$\underline{0.29\%}$	0.39%
∼10%		0.75%	$\underline{0.60\%}$	$\underline{0.45\%}$	0.52%

※ Underline shows the smaller error of two.

The errors are smaller for $a = 1.26$ than $a = 1.25$ in all cases with respect to $y = 2$. There was no a with smaller error in all cases with respect to $y = 3$, but the errors for the $a = 1.90$ is smaller than $a = 1.91$ in most cases. We set the rules for each ratio y including the case of $y = 2.5$ in Table 3.

Table 3: Rule of 'x' for installment savings

$y = 1.5$	$y = 2$	$y = 2.5$	$y = 3$
Rule of 76	Rule of 126	Rule of 162	Rule of 190

4. Calculating ratio from the rule of 'x'

We show the simple table for calculating ratio from the rule of 'x' in Table 4. These values are the ratios y calculated from Equation (3.2) with 40 years ($n = 40$) and 12 times of accumulation in a year($m = 12$).

Table 4: Simple table for calculating ratio from the rule of 'x' ($m = 12$, $n = 40$)

x	0	1	2	3	4	5	6	7	8	9
40	1.230	1.237	1.243	1.250	1.257	1.263	1.270	1.277	1.284	1.291
50	1.298	1.305	1.312	1.319	1.326	1.334	1.341	1.348	1.356	1.363
60	1.371	1.378	1.386	1.394	1.401	1.409	1.417	1.425	1.433	1.441
70	1.449	1.457	1.465	1.473	1.482	1.490	1.498	1.507	1.515	1.524
80	1.533	1.541	1.550	1.559	1.568	1.577	1.586	1.595	1.604	1.613
90	1.623	1.632	1.641	1.651	1.660	1.670	1.680	1.689	1.699	1.709
100	1.719	1.729	1.739	1.749	1.760	1.770	1.780	1.791	1.801	1.812
110	1.823	1.833	1.844	1.855	1.866	1.877	1.889	1.900	1.911	1.923
120	1.934	1.946	1.957	1.969	1.981	1.993	2.005	2.017	2.029	2.042
130	2.054	2.066	2.079	2.092	2.104	2.117	2.130	2.143	2.156	2.169
140	2.183	2.196	2.210	2.223	2.237	2.251	2.265	2.279	2.293	2.307
150	2.321	2.336	2.350	2.365	2.380	2.395	2.410	2.425	2.440	2.455
160	2.471	2.486	2.502	2.518	2.533	2.550	2.566	2.582	2.598	2.615
170	2.631	2.648	2.665	2.682	2.699	2.716	2.734	2.751	2.769	2.787
180	2.805	2.823	2.841	2.859	2.878	2.896	2.915	2.934	2.953	2.972
190	2.991	3.011	3.030	3.050	3.070	3.090	3.110	3.130	3.151	3.172
200	3.192	3.213	3.235	3.256	3.277	3.299	3.321	3.343	3.365	3.387
210	3.410	3.432	3.455	3.478	3.501	3.524	3.548	3.572	3.596	3.620
220	3.644	3.668	3.693	3.718	3.743	3.768	3.793	3.819	3.845	3.870
230	3.897	3.923	3.950	3.976	4.003	4.031	4.058	4.086	4.113	4.141

Example: y of the rule of 126 is 2.005, intersection of row 120 and column 6

Let me explain how to read this table. Suppose we want to find the value of the ratio y for the rule of 126 ($a = 1.26$). The value of y is 2.005, the intersection of row 120 and column 6 ($120 + 6 = 126$). The values change slightly depending on n and m, but you can neglect it because it is an approximation.

5. Concluding remarks

In this paper, we propose the "rule of 126" as a rule to double the money of installment savings or investments, corresponding to the rule of 72 to double the money of a one-time deposit or a lump-sum investment. If you knew such a simple rule for the installment savings deposit or investment, it would become more accessible. If you start savings at age 23 and keep it until you retire at age 65, the saving period is 42 years.

Using the rule of 126 where the future value becomes twice the principal, the interest rate required to achieve is $126/42 = 3(\%)$. Also, using the rule of 190 for tripling the money, the interest rate is $190/42 = 4.52(\%)$. Using the rule of 76 which multiplies the future value by 1.5 times, the required interest rate is $76/42 = 1.81(\%)$. The rule depends on "how much you want the ratio of future value to the total accumulated principal". Using the rule, you can easily and approximately obtain "how much you should assume the interest rate depending on the accumulation period" or. "how long you should assume the period depending on the interest rate".

References

[1] D. Luenberger, Investment Science, 2nd Edition, Oxford University Press, 2014.