Multi-period Optimization Model for Retirement Planning

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Abstract   We discuss a multi-period optimization model to obtain optimal investment, pension and consumption strategies for a household in retirement. There are many studies for retirement planning models in the literatures. Hibiki and Nishioka(2010) develop a multi-period optimization model for a couple in retirement in a simulated path approach, and derive optimal asset allocation, consumption and annuity using the life table modified by subjective health feeling. In this paper, we extend the model, and propose a multi-period stochastic programming model which provides the state-dependent asset allocation and consumption strategies. We employ mortality rates which are adjusted using Lee-Carter model and subjective health feeling. We solve a thirty-year problem to examine the characteristics of the model. We conduct the sensitivity analysis for mortality rate and inflation rate, and examine the impact of changing public pension system.

Keywords: finance, retirement planning, longevity risk, multi-period optimization

1.   Introduction

We discuss a multi-period optimization model to obtain optimal investment, pension and consumption strategies for a household in retirement. In recent years, lots of elderly people confer with financial planners. "Japan Association for Financial Planners" implemented the research in 2011 that 46.2% of their clients are 50 years old or older. They are exposed to several types of risks such as longevity risk, inflation risk, investment risk, interest rate risk and the risk of increasing medical expenses.

There are many studies for retirement planning models in the literatures. Milevsky and Young[19] examine the optimal annuitization, investment and consumption strategies of a utility-maximizing retiree, and obtain the analytical solutions for two annuitization strategies; gradual annuitization and complete switching. This is the first to integrate life annuity products into the portfolio choice. Milevsky, Moore and Young[18] derive the optimal investment and annuitization strategies for a retiree who wants to minimize the probability of lifetime ruin, and derive implicit analytical solutions of two cases of the free-boundary problems. Horneff, Maurer, Mitchell and Dus[9] compare three withdrawal rules (self-annuitization strategies), and the optimal withdrawal rule including optimal immediate annuitization and complete switching (deferring) at optimal switching age. Horneff, Maurer and Stamos[12] solve the problem to compute the optimal dynamic annuitization and asset allocation policy for a retiree with Epstein-Zin preferences, uncertain investment horizon, potential bequest motives, and pre-existing pension income. The optimal policies for gradual, partial and complete annuitization strategies are derived numerically by resorting to Gaussian quadrature integration. Horneff, Maurer, Mitchell and Stamos[13] model a dynamic utility maximizing investor who seeks to benefit from holding both equity and longevity insurance. Pang and Warshawsky[25] construct a discrete-time life-cycle model to

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derive optimal equity-bond-annuity portfolios for retired households with Epstein-Zin preferences. Hubener, Maurer and Rogalla[14] derive the optimal demand for stocks, bonds, annuities, and term life insurance for a retired couple with uncertainty in both lifetimes, using Gaussian quadrature integration. Gupta and Li[2] construct a multi-period optimization model of maximizing lifetime utility of consumption and wealth to decide how and when to purchase a lifetime annuity and the appropriate asset allocation. Samples are generated using a binomial tree asset-return model, and the sequential quadratic programming (SQP) methods is employed for solving large-scale optimization problems with nonlinear objectives and constraints. In addition, Gupta and Li[3] extend the optimization model in consideration of the bequest motive and the health evolution.

Hibiki and Nishioka[7] develop a multi-period optimization model for a couple in retirement who faces longevity risk using the simulated path model (Hibiki[4]), and derive optimal asset allocation, consumption and annuity using the life table modified by subjective health feeling. Using the simulated path approach, it is easy to describe stock return and mortality rate simultaneously, and to derive the optimal solutions for the model with practical constraints. However, the simulated path model adopts the fixed-mix strategy which is time-dependent, but not state-independent for asset allocation and consumption in order to describe the model easily. It is a drawback of this model.

In this paper, we extend the model by Hibiki and Nishioka[7]. We propose a multi-period stochastic programming model which involves determining state-dependent asset allocation and consumption strategies, and annuitization in retirement, and hedging risks as well. We solve the problem using the hybrid model in the simulated path approach proposed by Hibiki[5, 6].

The contributions and characteristics of our paper are in what follows.

(1) We develop the multi-period stochastic programming model for retirement planning in the simulated path approach. We determine asset allocation, annuitization and consumption strategies in retirement so that FP can give effective advices in the real world. We solve the problem using the hybrid model which allows conditional (state-dependent) decisions to be made. It is essentially different from Hibiki and Nishioka[7] from the viewpoint of modelling. We can determine the optimal strategies in practice under many possibilities in many years, specifically five decision nodes at each time and ten thousand simulated paths in 30 years in our paper.

(2) The amounts of consumption are divided into two types, usual consumption for living and additional consumption for richer life. It is assumed that the additional consumption is dependent on the amount of wealth, and the state-dependent and time-dependent function is proposed.

(3) Longevity risk is very important issue for a household in retirement, and therefore it is essential for each household to estimate individual mortality rate. It is said that subjective health feeling affects the mortality rate (Kaplan and Camacho[16], Mitoku, Takahashi and Hoshi[20]). We estimate the individual mortality rates which are adjusted using Lee-Carter model[17] and subjective health feeling.

We show Table 1 to compare our paper with previous studies, and clarify the differences.

This paper is organized as follows. In Section 2, we briefly explain the overview and important components in the retirement planning problem. In Section 3, we formulate the multi-period retirement planning model. In Section 4, we show the numerical analysis for a typical household, and we also conduct the sensitivity analysis. Section 5 provides our concluding remarks.

Table 1: Comparison of our paper with the previous studies

Authors	Year	Ref	Objective Function†1	Solution Method†2	Annuitization Strategy†3	Spouse	Bequest Motive	Mortality Rate	Subjective Health	Decision Variables		
										Stock	Consumption	Annuity
Bayraktar & Young	2007	[1]	SF	AS	—	No	No	constant	—	Yes	—	—
Milevsky, Moore & Young	2006	[18]	SF	AS PSOR	CS(τ)	No	No	constant /Gompertz	Yes	Yes	—	—
Milevsky & Young	2007	[19]	CRRA	AS	GA(t) CS(τ)	No	Yes	life table (Gompertz)	Yes	Yes	Yes	Yes
Horneff, Maurer & Stamos	2008	[11]	EZ	GQ	GA(t)	No	Yes	life table (Gompertz)	Yes	Yes	Yes	Yes
Horneff, Maurer & Stamos	2008	[12]	EZ	GQ	GA(t) PS(τ), CS(τ)	No	Yes	life table (2 types)	Yes	Yes	Yes	Yes
Horneff, Maurer, Mitchell & Dus	2008	[9]	CRRA	GQ	PS(0), CS(τ)	No	Yes	life table (2 types)	Yes	Yes	—	Yes
Horneff, Maurer, Mitchell & Stamos	2010	[13]	CRRA	GQ	GA(t)	No	Yes	life table	No	Yes	Yes	Yes
Horneff, Maurer & Rogalla	2010	[10]	CRRA	GQ	GA(T_s)	No	Yes	life table (Lee-Carter)	No	Yes	Yes	Yes
Hubener, Maurer & Rogalla	2014	[14]	CRRA	GQ	GA(t)	Yes	Yes	life table (3 types)	No	Yes	Yes	Yes
Pang & Warshawsky	2010	[25]	EZ	GQ	GA(t)	No	Yes	De Nardi et al.(2006)	Yes	Yes	Yes	Yes
Gupta & Li	2007	[2]	CRRA	SQP	GA(T_s)	No	No	life table	No	Yes	Yes	Yes
Gupta & Li	2013	[3]	CRRA	SQP	GA(T_s)	No	Yes	health evolution model	Yes	Yes	Yes	Yes
Hibiki & Nishioka	2010	[7]	LPM	LP	PS(0)	Yes	Yes	life table (subjective health)	Yes	Yes	Yes	Yes
Our paper	2015		LPM	NLP	PS(0)	Yes	Yes	life table (Lee-Carter & subjective health)	Yes	State-Dependent		Yes

†1 Objective Function : CRRA: CRRA utility, SF: ShortFall probability, EZ utility: Epstein-Zin utility, LPM: Lower Partial Moment

†2 Solution Method : AS: Analytical Solution, GQ: Gaussian Quadrature integration, SQP: Sequential Quadratic Programming, LP: Linear Programming, NLP: Non-LP, PSOR: Projected Successive OverRelaxation method

†3 Annuitization Strategy: GA(t): Gradual Annuitization, paid at time t and received after time $t + 1$ for any time t, GA(T_s): GA, paid before time T_s and received after time $T_s + 1$, PS(τ): Partial Switching only once at time τ, CS(τ): Complete Switching only once at time τ, SA: Self-Annuitization

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2. Retirement planning problem

2.1. Overview

We define a household as a family composed of a householder and a spouse. Suppose the householder is just retired at 65 years old, and the spouse is also 65 years old. Planning period is from now (retirement of householder) to time when both die, and it is assumed that a maximum period is 30 years. One period is a year, and therefore we formulate the model for a 30-year problem. In addition to public pension annuity for family income, and minimum living cost, medical expenses, and planned consumption for expenditures, we obtain asset return and private pension annuity, and pay additional consumption, which are derived as the optimal strategies. The objective function is defined as the sum of the expected additional consumption for richer life and the expected amount of wealth required for bequest motive minus expected shortfall from the target required to manage longevity risk. We calculate the present value of the amounts of wealth at the time when two family members die. The problem is solved so that it can be maximized in the simulated path approach.

At first, we generate simulated paths for random time of death, medical expenses, risk-free rate, prices of risky assets, and inflation rate by using Monte Carlo method. Second, we solve the stochastic programming problem to determine the optimal asset mix (risky assets and risk-free asset), additional consumption rate, and private pensions for a householder and a spouse by using sophisticated mathematical programming software. We show the summary in Figure 1.
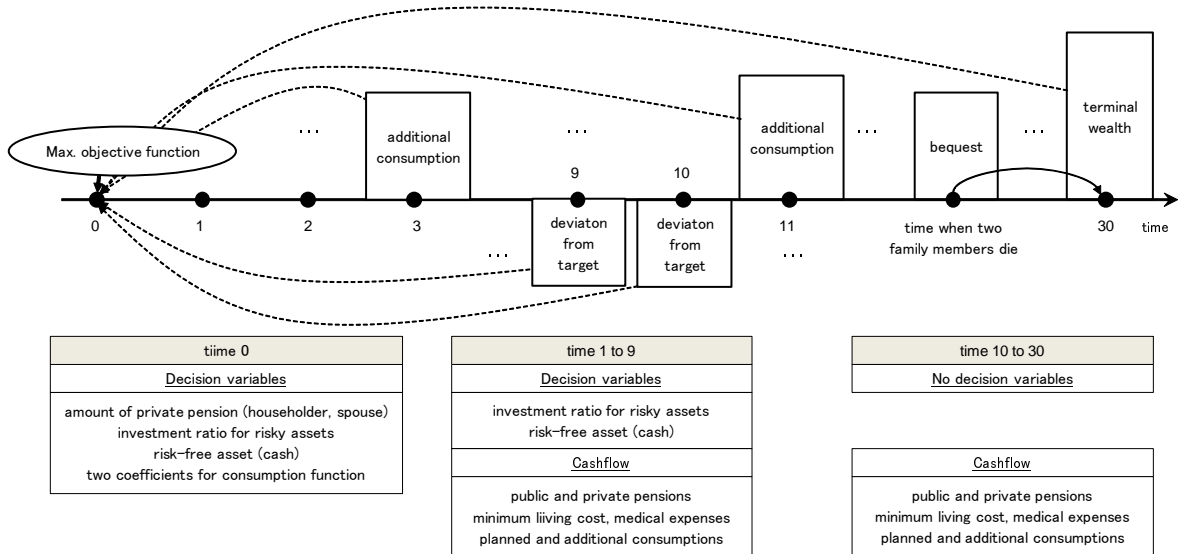

Figure 1: Planning period and decision variables

2.2. Mortality rate

We need to employ the mortality rate associated with the increase in length of life and individual health conditions in order to evaluate longevity risk which is critical for a household in retirement. In this paper, we generate the life table where these can be reflected explicitly. Specifically, we estimate the future mortality rates using Lee-Carter model, and generate the dynamic life table. In addition, we modify the dynamic life table using subjective health feeling estimated by Hibiki and Nishioka[7]. Details are as follows.

2.2.1. Lee-Carter model

We forecast the future mortality rates of Japanese using a well-known Lee-Carter method in order to generate the dynamic life table involving the calendar year effect explicitly. Lee and Carter[17] formulate the log-mortality rate of age x at time t, as

$$\ln m_{xt} = a_x + b_x k_t + \varepsilon_{xt} \quad (x = 0, \ldots, \omega; \ t = 1, \ldots, T), \qquad (2.1)$$

and estimate the three kinds of parameters using historical data, where ω is a maximum age in the historical data, and ε_{xt} is a random error. The parameter a_x shows the log-mortality rate of age x which is independent of the calendar year, k_t shows the calendar year effect to the mortality rate which is dependent on time t, and b_x shows the sensitivity of the mortality rate of age x for the parameter k_t.

We estimate three kinds of parameters in Equation (2.1), using the mortality rate from 1970 to 2010 in the Japanese Mortality Database of the National Institute of Population and Social Security Research[24]. We apply ARIMA(p, d, q) to $\hat{k}_t$, estimates of k_t, and select the models by AIC. As the results, ARIMA(1,1,2) model is selected for both male and female, as follows.

$$\hat{k}_t - \hat{k}_{t-1} = \beta_1(\hat{k}_{t-1} - \hat{k}_{t-2}) + a_t + \gamma_1 a_{t-1} + \gamma_2 a_{t-2} + e_t$$

Estimates of ARIMA(1,1,2) are shown in Tables 2.

Table 2: Estimates of ARIMA(1,1,2)

	$\hat{\beta}_1$	$\hat{\gamma}_1$	$\hat{\gamma}_2$	σ_e
male	0.9941	-1.3649	0.5052	2.1269
female	0.9915	-1.3675	0.5385	2.2650

Based on the estimates, we forecast the future mortality rates and generate the dynamic life tables. We show the survival rate functions calculated by the dynamic life tables for the person who is 65 years old in 2010 in Figure 2. We also show the rates by the static life table in 2010 for comparison.
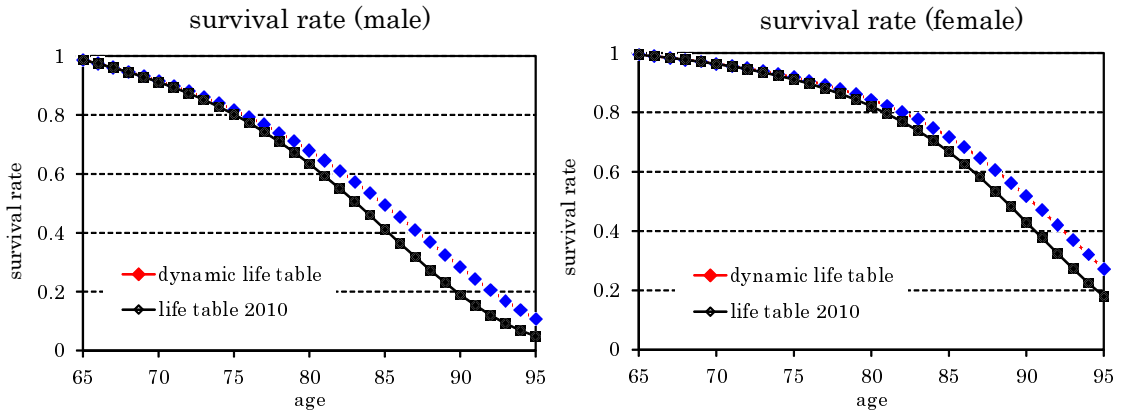

Figure 2: Survival rate function for male on the left side and female on the right side

The survival rates calculated by the dynamic life tables are larger than those by the static tables. We find that the mortality rates tend to decrease for both male and female, and the longevity risk increases in the future.

2.2.2. Subjective health feeling

Subjective health feeling is the index evaluated subjectively and personally for the health condition. This is recognized as one of the new health index, and it can show the qualitative aspect of the health condition, which cannot be expressed by the objective health index such as mortality rate, morbidity, and so on. There are a lot of studies which show that the subjective health feeling relates to the mortality rate (Kaplan and Camacho[16], Mitoku, Takahashi and Hoshi[20]). Hibiki and Nishioka [7] analyze the three waves of data (wave1(1987), wave2(1990) and wave3(1993)) from the National Survey of Japanese Elderly conducted jointly by the Institute of Gerontology at the University of Michigan and the Tokyo Metropolitan Institute of Gerontology. The data includes subjective health feeling and disease, and the following results are shown.

(1) The survival rates of male are dependent on the subjective health feeling and disease. If male has no disease, the survival rates are independent on subjective health feeling. But, if male has disease, subjective health feeling affects the survival rates. Therefore, we modify the 20th life tables (2010) prepared by Ministry of Health, Labour and Welfare [21] in Japan for four kinds of the conditions; 'excellent or very good with disease', 'good or fair with disease', 'poor with disease', 'no disease'.

(2) The survival rates of female are dependent on subjective health feeling, regardless of disease. We modify the 20th life tables (2010) for three kinds of the conditions; 'excellent, very good or good', 'fair', 'poor'.

In this paper, we modify the dynamic life table generated by Lee-Carter for each subjective health feeling. We show the survival rates in Figure 3.
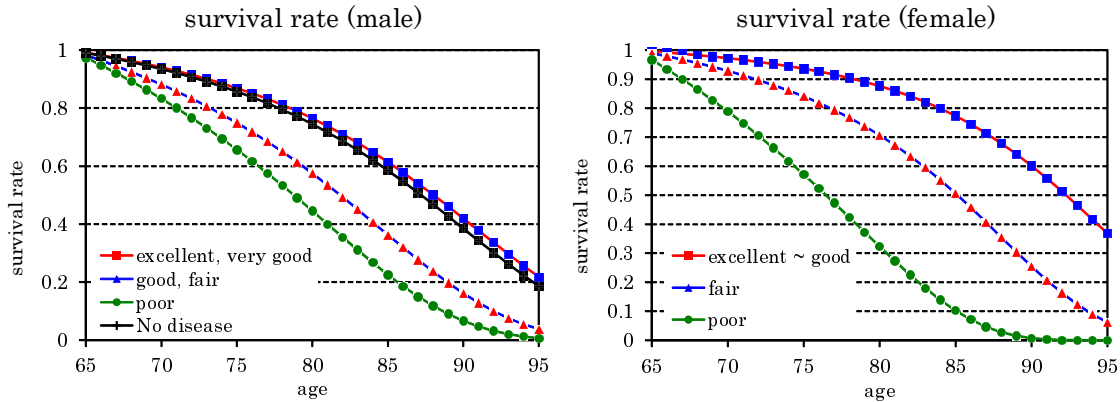

Figure 3: Survival rates based on the dynamic life table and subjective health feeling

We can generate the dynamic life table in consideration of subjective health feeling.

2.3. Income from pension

The income of the retired household consists of public pension and private pension for a householder and a spouse.

2.3.1. Public pension

Public pension consists of national pension (old age basic pension) and employees' pension insurance. We assume these amounts are given. The macro-economic indexation or system called macro-economic slide formula is introduced from 2004 in Japanese public pension system.

The amount of pension is adjusted on the basis of the following pension revision rate.

pension revision rate

$$= \begin{cases} \max(\text{inflation rate} - \text{indexation adjustment rate}, 0) & \text{for positive inflation rate} \\ \text{inflation rate} & \text{for negative inflation rate} \end{cases}$$

The indexation adjustment rate is defined as the sum of the decrease rate in the number of insured persons of the entire public pension and the fixed rate after the growth rate of the average pension benefit period (average life expectancy). It is 0.9%, based on the expectation up to 2025.

2.3.2. Private pension

We use a life annuity with payment guaranteed in ten years sold by "Zenrosai"(National Federation of Workers and Consumers Insurance Cooperatives) as a private pension plan to examine the model in practice. We pay the premium in a lump sum at time 0 (65 years old), and receive the annuity from time 1 (66 years old). The maximum annuity is 0.9 million yen each year, and the premium for male is about 16.94 million yen, and the premium for female is about 20.93 million yen.

2.4. Expenditures

The expenditures of the retired household consist of minimum living cost, medical expenses, planned consumption, and additional consumption for richer life.

(1) Minimum living cost

We estimate the minimum living cost C_d with public pension benefit P, using 'National survey of family income and expenditure in 2009' by Statistic Bureau, Ministry of Internal Affairs and Communications[26] as follows.

$$C_d = 98.99 + 0.6464P, \quad R^2 = 0.988$$

It is assumed that a household can keep a usual consumption level if both are alive, however a consumption level must be κ_1 times the level C_d if one of family members dies, where κ_1 is a parameter associated with a consumption level. For example, we set $\kappa_1 = 0.7$ when it has to allow for the 70% consumption level. The value of $\kappa_1 = 0.7$ is used in the numerical analysis of Section 4.

(2) Medical expenses

The medical expenses of each age are estimated using 'National Medical Care Expenditure (2012)' by Ministry of Health, Labour and Welfare[23]. We assume the medical expense follows a log-normal distribution, and they are autocorrelated between time $t - 1$ and time t. The medical expenses are dependent on the self-pay ratio, which is 30% under 70 years old, 20% between 70 and 74 years old, and 10% over 75 years old.

(3) Planned consumption

The planned consumption is the expenditure expected at time 0, such as cost for annual event, celebration money for entering a school.

(4) Additional consumption for richer life

The additional consumption is allowed for richer life except minimum living cost, medical expenses, and planned consumption. We assume the retired household wants to increase it.[1] Therefore, we can determine the expenditures for additional consumption, which are

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[1]As well as the minimum living cost, we assume the additional consumption is reduced to κ_2 times the consumption if one of family members dies. The value of $\kappa_2 = 1$ is used in the numerical analysis of Section 4.

included in the objective function to be maximized. However, they should be dependent on the amount of wealth. We formulate the model in Section 3.2.

2.5. Asset return

In this paper, we assume a retired household invests in J risky assets and a risk-free asset. We use one-year time deposit for the risk-free asset. The one-year deposit rate is highly correlated with six-month LIBOR. Using the historical data from January 2004 to December 2013, the linear relationship between the one-year deposit rate y and the six-month LIBOR x can be described by the regression analysis as

$$y = -0.0308 + 0.4132x, \quad R^2 = 0.904. \tag{2.2}$$

We generate samples of six-month LIBOR using Hull-White model [15].[2] We calculate samples of the one-year deposit rate by Equation (2.2).

We assume the investment in risky assets are limited within T_R year from 65 years old, and therefore the household has only risk-free asset after T_R year. We forecast the rates of return of risky assets using the building block method. Specifically, we separate them into two components; risk-free rate and risk premium as

$$\mu_{jt} = r_{t-1} + RP_{jt} \ (t = 1, ..., T_R), \tag{2.3}$$

where μ_{jt} is a rate of return of asset j in period t, r_{t-1} is interest rate in period t (determined at time $t-1$), and RP_{jt} is a risk premium of asset j in period t. We assume the risk premium is normally distributed for simplicity. We generate samples of risk premiums using the Monte Carlo method, and we calculated the rates of return of risky assets by adding them to the interest rates.

3. Multi-period optimization model for retirement planning

3.1. Hybrid model and conditional decision

Hibiki[5, 6] develop the hybrid model in the simulated path approach, which allows conditional decisions to be made for the similar states bundled at each time using the sample returns generated by the Monte Carlo method.

We employ the lattice structure as the modeling structure with respect to the decision nodes in this paper. We call the hybrid model with m decision nodes at each time 'hybrid Nm model' hereafter. As examples of the lattice structure, we depict the hybrid N5 model on the left-hand side of Figure 4. The number of paths is twelve and the same decision is made for paths in each node at each time.

An investment unit function is used in the hybrid model in order to express the decision rule which is defined to satisfy the non-anticipativity condition. Let $W_t^{(i)}$ denote the amount of wealth of time t and path i, and $\rho_{jt}^{(i)}$ denote the price of risky asset j of time t and path i. Using them, it is defined as Equation (3.1) which expresses the path-dependent investment unit for path i using the decision variable of the investment proportion z_{jt}^s for asset j, time

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[2]Hull-White model is one of the well-known interest rate model, and it is described as

$$dr_t^L = \left[\theta_t - ar_t^L\right] dt + \sigma dz_t$$

where r_t^L is the short rate, a is the mean reversion rate, σ is the standard deviation of the short rate, and θ_t is a function of time chosen to ensure that the model fits the initial term structure. The short rate reverts to θ_t/a at rate a.
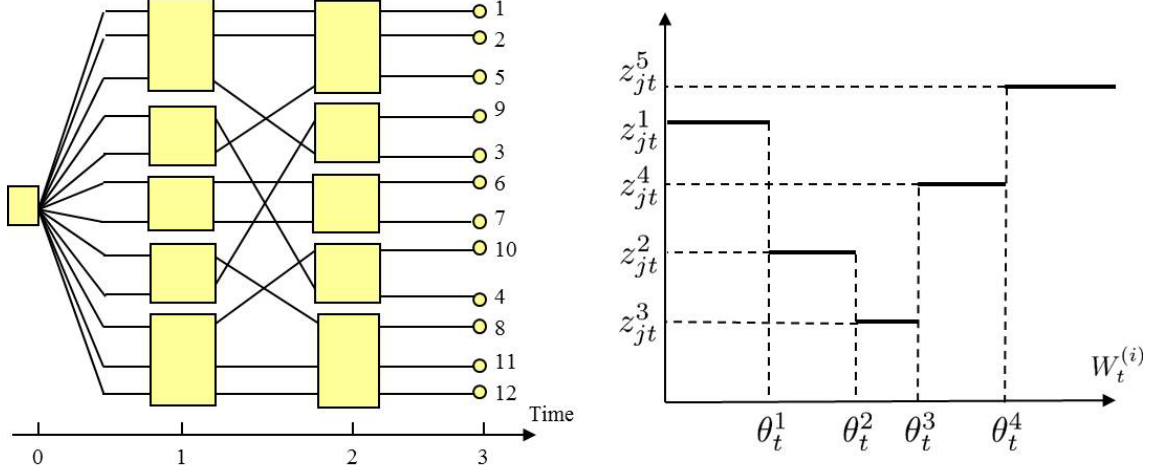
Figure 4: Hybrid model structure and step function for hybrid N5 model

t and decision node s. The assigned node is dependent on the amount of wealth as in Equation (3.2). θ_t^u is a threshold which separates node u and $u+1$ at time t.

$$h^{(i)}(z_{jt}^s) = \left(\frac{W_t^{(i)}}{\rho_{jt}^{(i)}}\right) z_{jt}^s \tag{3.1}$$

$$s = \begin{cases} 1 & (W_t^{(i)} \leq \theta_t^1) \\ u & (\theta_t^{u-1} \leq W_t^{(i)} \leq \theta_t^u, u = 2, \ldots, m-1) \\ m & (W_t^{(i)} \geq \theta_t^{m-1}) \end{cases}$$
$$(j = 1, \ldots, J; \, t = 1, \ldots, T-1) \tag{3.2}$$

We show a step function for the hybrid N5 model on the right-hand side of Figure 4.

3.2. State-dependent and time-dependent consumption

As shown in Section 2.4, a household in retirement would like to increase additional consumption for richer life. In this paper, we formulate the time-dependent consumption rate to wealth as,

$$C_{\alpha,t} = \begin{cases} \dfrac{C_{\alpha,y} - C_{\alpha,1}}{y-1}(t-1) + C_{\alpha,1} & (t \leq y) \\ C_{\alpha,y} & (t \geq y) \end{cases}, \tag{3.3}$$

$$0 \leq k_L \leq \frac{C_{\alpha,1}}{C_{\alpha,y}} \leq k_U, \tag{3.4}$$

where y is a kinked point of the function, $C_{\alpha,1}$ and $C_{\alpha,y}$ are additional consumption rates at time 1 and y, and k_L and k_U are lower and upper multiple bounds. We show the example function in Figure 5.

We propose the state-dependent consumption function where the amounts of additional consumption are proportional to the amount of wealth as,
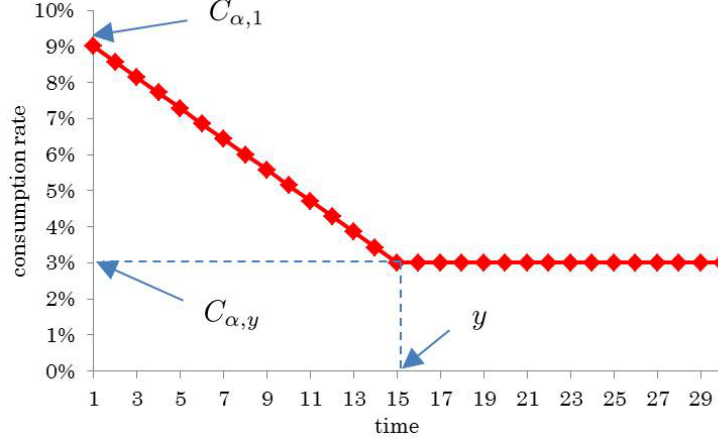
$$C_{\alpha,t}^{(i)} = W_t^{(i)} C_{\alpha,t}. \tag{3.5}$$

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Figure 5: time-dependent consumption rate

3.3. Formulation

(1) Notations

① Sets

S_t : set of fixed-decision nodes at time t, $(t = 1, \ldots, T_R - 1)$. T_R is the investment period for risky assets.

V_{t-1}^s : set of paths passing any node s at time $t-1$, $(t = 2, \ldots, T_R; s \in S_{t-1})$

② Parameters[3]

J : number of risky assets

T : number of periods

I : number of sample paths

$\tau_{AM,t}^{(i)}$: one if a householder is alive on path i at time t and zero otherwise.

$\tau_{LM,t}^{(i)}$: one if a householder dies on path i at time t and zero otherwise

$\tau_{AF,t}^{(i)}$: one if a spouse is alive on path i at time t and zero otherwise

$\tau_{LF,t}^{(i)}$: one if a spouse dies on path i at time t and zero otherwise

$\tau_{A,t}^{(i)}$: one if any family member is alive on path i at time t, and zero otherwise

$$\tau_{A,t}^{(i)} = \tau_{AM,t}^{(i)} + \tau_{AF,t}^{(i)} - \tau_{AM,t}^{(i)} \times \tau_{AF,t}^{(i)}$$

ρ_{j0} : price of risky asset j at time 0

$\rho_{jt}^{(i)}$: price of risky asset j on path i at time t

r_0 : interest rate in period 1 or at time 0.

$r_{t-1}^{(i)}$: interest rate on path i in period t or at time $t-1$ $(t = 2, \ldots, T)$.

df_t : discount factor used for cashflow at time t

W_0 : initial amount of wealth

$P_t^{(i)}$: disposable income on path i at time t (except private pension)

$C_{d,t}^{(i)}$: expenditures for living on path i at time t (except medical expense)

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[3]In the case that $(j = 1, \ldots, J)$ is used for asset j, $(t = 1, \ldots, T)$ for time t, and $(i = 1, \ldots, I)$ for path i, these descriptions are omitted.

$H_t^{(i)}$: medical expense on path i at time t

$C_{p,t}^{(i)}$: planned consumption on path i at time t

y : kinked point of the function for additional consumption for richer life

k_L, k_U : lower and upper multiple bounds of additional consumption

A_M, A_F : private pension premiums per unit of a household and a spouse at time 0

a_M, a_F : private pension payments per unit of a household and a spouse at time 0

T_g : guaranteed payment period of private pension

b : strength of bequest motive

γ : risk aversion

$\omega_{R,t}$: weight associated with risk at time t where $\sum_{t=1}^{T} \omega_{R,t} = 1$

L_v : lower bound of average cash

$W_{G,t}$: target wealth at time t

③ Decision variables

D_0^- : Cash outflow at time 0

$D_t^{+(i)}$: Cash inflow on path i at time t

$D_t^{-(i)}$: Cash outflow on path i at time t

$D_t^{(i)}$: Net cash flow on path i at time t

$W_t^{(i)}$: wealth on path i at time t

$C_{\alpha,t}^{(i)}$: additional consumption for richer life on path i at time t

z_{j0} : investment proportion of risky asset j at time 0

z_{jt}^s : investment proportion of decision node s, risky asset j and time t ($t = 1, \ldots, T_R - 1$; $s \in S_t$).

x_M, x_F : numbers of units of private pension for a householder and a spouse at time 0

$C_{\alpha,1}$, $C_{\alpha,y}$: additional consumption rates at time 1 and time y

$q_t^{(i)}$: shortfall from target wealth on path i at time t

(2) Formulation

We construct the stochastic programming model in the simulated path approach. As explained in Section 2.1, the objective function consists of three terms; expected present value (PV) of bequest (wealth), expected PV of additional consumption, and expected PV of shortfall from target wealth. We have the tradeoff between bequest and consumption, and they are weighted by the bequest motive. We also have the tradeoff between the sum of bequest and consumption and the shortfall, and they are weighted by the risk aversion. We solve the problem in consideration of these tradeoffs. The formulation is as follows.

$$\textbf{Maximize} \quad b \times \frac{1}{I}\sum_{i=1}^{I} df_T W_T^{(i)} + (1-b) \times \frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T} df_t C_{\alpha,t}^{(i)}$$

$$-\gamma \left\{ \frac{1}{I}\sum_{i=1}^{I}\sum_{t=1}^{T} \tau_{A,t}^{(i)} \omega_{R,t} df_t q_t^{(i)} \right\} \tag{3.6}$$

subject to

$$D_0^- = A_M x_M + A_F x_F \tag{3.7}$$

$$D_t^{+(i)} = \begin{cases} P_t^{(i)} + a_M x_M + a_F x_F & (t=1,...,T_g) \\ P_t^{(i)} + \tau_{AM,t}^{(i)} a_M x_M + \tau_{AF,t}^{(i)} a_F x_F & (t=T_g+1,...,T) \end{cases}, (i=1,\ldots,I) \tag{3.8}$$

$$D_t^{-(i)} = C_{d,t}^{(i)} + H_t^{(i)} + C_{p,t}^{(i)} + C_{\alpha,t}^{(i)} \ (t=1,\ldots,T; i=1,\ldots,I) \tag{3.9}$$

$$D_t^{(i)} = D_t^{+(i)} - D_t^{-(i)} \ (t=1,\ldots,T; i=1,\ldots,I) \tag{3.10}$$

$$\sum_{j=1}^{J} \rho_{j0} z_{j0} + D_0^- \leq W_0 \tag{3.11}$$

$$W_1^{(i)} = \sum_{j=1}^{J} \left\{ \rho_{j1}^{(i)} - (1+r_0)\rho_{j0} \right\} z_{j0} + (1+r_0)(W_0 - D_0^-) + D_1^{(i)} \ (i=1,\ldots,I) \tag{3.12}$$

$$W_t^{(i)} = \tau_{A,t-1}^{(i)} \sum_{j=1}^{J} \left\{ \rho_{jt}^{(i)} - (1+r_{t-1}^{(i)})\rho_{j,t-1}^{(i)} \right\} h^{(i)}(z_{j,t-1}^s) + (1+r_{t-1}^{(i)})W_{t-1}^{(i)} + D_t^{(i)}$$

$$(t=2,\ldots,T_R; s\in S_{t-1}; i\in V_{t-1}^s) \tag{3.13}$$

$$W_t^{(i)} = (1+r_{t-1}^{(i)})W_{t-1}^{(i)} + D_t^{(i)} \ (t=T_R+1,\ldots,T; i=1,\ldots,I) \tag{3.14}$$

$$W_t^{(i)} + q_t^{(i)} \geq W_{G,t} \ (t=1,\ldots,T; i=1,\ldots,I) \tag{3.15}$$

$$h^{(i)}(z_{jt}^s) = \left(\frac{W_t^{(i)}}{\rho_{jt}^{(i)}} \right) z_{jt}^s \ (j=1,\ldots,J; s\in S_t; i\in V_t^s) \tag{3.16}$$

$$z_{j0} \geq 0 \ (j=1,\ldots,J) \tag{3.17}$$

$$z_{jt}^s \geq 0 \ (j=1,\ldots,J; t=1,\ldots,T_R-1; s\in S_t) \tag{3.18}$$

$$\sum_{j=1}^{J} \rho_{j0} z_{j0} \leq (1-L_v)(W_0 - D_0^-) \tag{3.19}$$

$$\sum_{i=1}^{I} \rho_{jt}^{(i)} h^{(i)}(z_{jt}^s) \leq (1-L_v)\sum_{i=1}^{I} W_t^{(i)} \ (t=1,\ldots,T_R-1) \tag{3.20}$$

$$W_T^{(i)} \geq 0 \ (i=1,\ldots,I) \tag{3.21}$$

$$q_t^{(i)} \geq 0 \ (t=1,\ldots,T; i=1,\ldots,I) \tag{3.22}$$

$$0 \leq k_L \leq \frac{C_{\alpha,1}}{C_{\alpha,y}} \leq k_U \tag{3.23}$$

$$0 \leq x_M \leq 1 \tag{3.24}$$

$$0 \leq x_F \leq 1 \tag{3.25}$$

Equation (3.7) is cash outflow at time 0 which consists of private pension premiums. Equation (3.8) is cash inflow at time t which consists of disposable incomes. Even if both family

members die, we assume the household obtains private pensions in the guaranteed payment period. Equation (3.9) is cash outflow at time t which consists of four kinds of expenditures, shown in Section 2.4. Equation (3.10) is net cash flow at time t. Equation (3.11) is a budget constraint at time 0. Equations (3.12) and (3.13) are the calculations of the amount of wealth at time t derived by the investment in risky assets and a risk-free asset at time $t-1$ until time T_R. We calculate the amounts of wealth, assuming we invest in a risk-free asset after both family members die. Equation (3.14) is the calculation of the amount of wealth at time t derived by the investment in a risk-free asset after time $T_R + 1$. Equation (3.15) is used to calculate LPM. The shortfall $q_t^{(i)}$ is calculated based on $W_t^{(i)}$ for all t, but the shortfall after both family members die is not evaluated in the expected shortfall in the objective function as shown in the third term of Equation (3.6). Equations (3.17) and (3.18) are non-negativity constraints for risky assets. Equations (3.19) and (3.20) are lower bound constraints for a risk-free asset. Equation (3.21) is a non-negativity constraint for the amount of terminal wealth. Equation (3.23) is a constraint for the consumption function. Equations (3.24) and (3.25) are lower and upper bound constraints for private pensions.

3.4. Iterative algorithm

The weights of the risky assets are expressed by the state-dependent and step function of the amount of wealth as in Equation (3.1), and the additional consumption is also expressed by the state-dependent and time-dependent function as in Equation (3.5). However, it is difficult to solve the problem because it is formulated with the non-linear and non-convex function. Therefore, we solve the problem using the modification of iterative algorithm developed by Hibiki [6].

If the amounts of wealth are parameters in Equation (3.1) and (3.5), the original problem can be reduced to a LP problem. Then we replace the values derived in the previous iteration into the unknown variables, and solve the LP problem iteratively. The algorithm has the following six steps.

Step 1: We set up $h^{(i)}(z_{jt}) = z_{jt}$ as the investment unit function and $C_{\alpha,t}^{(i)} = C_{\alpha,t}$ as the consumption function. We solve a problem using the hybrid N1 model with the fixed-unit strategy.[4] Let Obj_0 denote the objective function value, and set $k = 1$.

Step 2: Let $W_{t(k-1)}^{(i)*}$ be the amount of wealth of path i at time t, and we calculate it. We set up $h^{(i)}(z_{jt}) = \left(\frac{W_{t(k-1)}^{(i)*}}{\rho_{jt}^{(i)}}\right) z_{jt}$ as the investment unit function, and $C_{\alpha,t}^{(i)} = W_{t(k-1)}^{(i)*} C_{\alpha,t}$ as the consumption function at the k-th iteration. We solve the problem using the hybrid N1 model with the fixed-proportion strategy. We calculate the objective function value Obj_k.

Step 3: Go to Step 4 if a value $Obj_k - Obj_{k-1}$ is lower than a tolerance. Otherwise, set $k \leftarrow k + 1$, and return to Step 2.

Step 4: Set $k \leftarrow k + 1$. $W_{t(k-1)}^{(i)*}$ is calculated as the amount of wealth of path i at time t of

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[4]We replace Equations (3.16) and (3.18) with Equation (3.26) for the hybrid N1 model with the fixed-unit strategy in Step 1, and Equation (3.27) for the hybrid N1 model with the fixed-proportion strategy in Step 2, respectively.

$$[\text{Unit (N1)}] \qquad h^{(i)}(z_{jt}) = z_{jt} \geq 0, \ (j = 1, \ldots, J; \ t = 1, \ldots, T-1), \tag{3.26}$$

$$[\text{Proportion (N1)}] \qquad h^{(i)}(z_{jt}) = \left(\frac{W_t^{(i)}}{\rho_{jt}^{(i)}}\right) z_{jt}; \ z_{jt} \geq 0, \ (j = 1, \ldots, J; \ t = 1, \ldots, T-1). \tag{3.27}$$

We also replace $C_{\alpha,t}^{(i)}$ with $C_{\alpha,t}$ in Equations (3.6), and (3.9).

the $(k-1)$th iteration. We sort the amounts of wealth in ascending order, and divide them into m nodes at each time. θ_t^{u*} is the point which separates two nodes of u and $u+1$, and determined as the boundaries of the amounts of wealth, using $W_{t(k-1)}^{(i)*}$.

Step 5: We solve the problem using the following Equations.

$$h^{(i)}(z_{jt}^s) = \left(\frac{W_{t(k-1)}^{(i)*}}{\rho_{jt}^{(i)}}\right) z_{jt}^s \tag{3.28}$$

$$s = \begin{cases} 1 & (W_{t(k-1)}^{(i)*} \leq \theta_t^{1*}) \\ u & (\theta_t^{u-1*} \leq W_{t(k-1)}^{(i)} \leq \theta_t^{u*}, u = 2, \ldots, m-1) \\ m & (W_{t(k-1)}^{(i)} \geq \theta_t^{m-1*}) \end{cases}$$
$$(j = 1, \ldots, J; t = 1, \ldots, T-1). \tag{3.29}$$

$$C_{\alpha,t}^{(i)} = W_{t(k-1)}^{(i)*} C_{\alpha,t} \tag{3.30}$$

We calculate the objective function value Obj_k and the amount of wealth $W_{t(k)}^{(i)*}$ using the optimal solutions.

Step 6: Stop if a value $Obj_k - Obj_{k-1}$ is lower than a tolerance. Otherwise, set $k \leftarrow k+1$, and return to Step 5.

The algorithm does not guarantee to derive the global optimal solutions, as stated in Hibiki [6]. Moreover, the local optimality is not also theoretically guaranteed, but empirically the objective function value almost converges.[5] This algorithm is a heuristic one.

4. Numerical analysis

We test numerical examples for a hypothetical household which has no bequest motive $(b = 0)$. The household invests in a risky asset (stock) and a risk-free asset (cash). The householder has a disease, and his subjective health feeling is 'good'. The subjective health feeling of a spouse is 'excellent'. Both a householder and a spouse can purchase a life annuity with payment guaranteed in ten years sold by "Zenrosai" as shown in Section 2.3.2. We use the base parameters in Table 3.

We conduct the analysis using the hybrid N5 model for four cases as follows.

Case 1. Effect of purchasing the private pension

Case 2. Sensitivity of mortality rate

Case 3. Sensitivity of inflation rate

Case 4. Impact of changing the public pension system

All of the problems are solved using NUOPT (Ver. 14.1) — mathematical programming software package developed by NTT DATA Mathematical System, Inc. on Windows 7 personal computer which has Corei7-2675QM 2.2 GHz CPU and 16 GB memory.

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[5]Empirically, the objective function value almost converges after three iterations in both Step 2 and Step 5 in the numerical analysis of Section 4. Therefore we conduct the analysis fixing the number of iterations instead of the convergence condition in Steps 2 and 5. The objective function value in Hibiki [6] converges after two iterations. The iterative algorithm is also used in the multi-period optimal asset allocation problem of minimizing CVaR (Hirano and Hibiki[8]), and the objective function value converges as well. We find this is the stable algorithm to derive the solution.

Table 3: Base parameters

Parameters	values
expected rate of return of stock	$\mu_s = 2.5\%$
standard deviation of rate of return of stock	$\sigma_s = 20\%$
initial amount of wealth(million yen)	$W_0 = 20$
target wealth at time t (million yen)	$W_{G,t} = 20 - 0.66t$, $W_{G,T} = 0.2$
planning period, investment period(year)	$T = 30$, $T_R = 10$
lower bound of average cash	$L_v = 10\%$
private pension payment per unit (million yen)	$a_M, a_F = 0.9$
private pension premium per unit(million yen)	$A_M = 16.94$, $A_F = 20.93$
guaranteed payment period of private pension(year)	$T_g = 10$
kinked point of consumption function (year)	$y = 15$
multiple bounds of additional consumption	$k_L = 1$, $k_U = 3$
risk aversion	$\gamma = 10$
weight associated with risk	$\omega_{R,t} = \frac{1}{T-t+1}/\sum_{k=1}^{T} \frac{1}{T-k+1}$
inflation rate	$f = 0\%$
number of paths	$I = 10,000$

4.1. Case 1: Effect of purchasing the private pension

We solve the problems with private pension and without it in order to test the effects. We show the two kinds of efficient frontiers on the left-hand side of Figure 6.
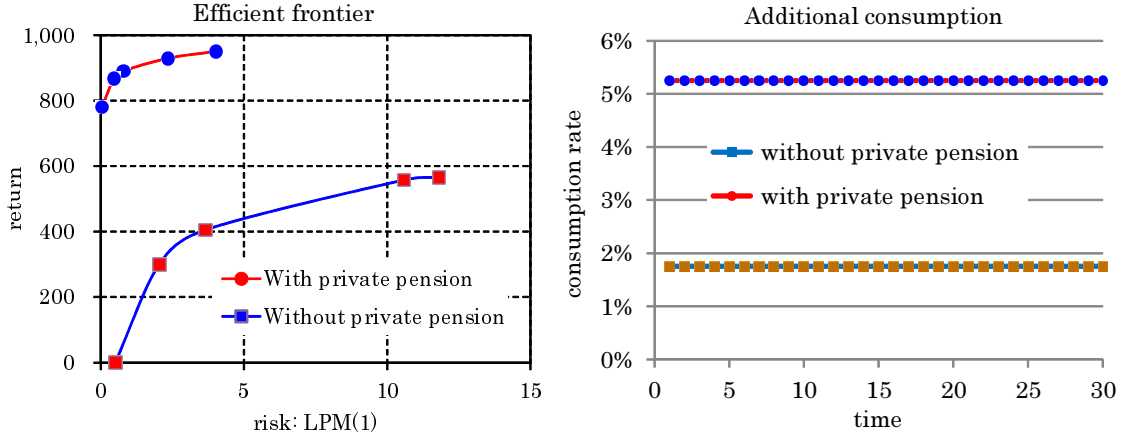

Figure 6: Efficient frontier and consumption

The vertical axis ('return') shows the second term of the objective function due to $b = 0$, and the horizontal axis ('risk:LPM(1)') shows the value of the third term except γ. The efficient frontier shifts to upper left by joining the private pension plan. The reason is that family members can increase the additional consumption for richer life and avoid shortfall risk, owing to getting income from private pension.

We show the optimal additional consumption rate on the right-hand side of Figure 6. The additional consumption rate increases by purchasing the private pension. The reason is a household can increase the stable income from the private pension at each time, and spend more money for additional consumption than a household without the private pension.

The constant consumption strategy is optimal regardless of the pension strategies, or $C_{\alpha,1} = C_{\alpha,y}$ which means the multiple of $C_{\alpha,1}$ to $C_{\alpha,y}$ is equal to the lower bound($k_L = 1$).

The reason is that the longevity risk increases if the household spends more money during the early period than the later period, and the household needs to reduce it.

We solve the problems with the private pension for four kinds of lower bounds ($k_L = 0.3, 0.5, 0.7, 1$) in order to examine the effect to the objective function values and the optimal additional consumption rates. We show the results in Figure 7.
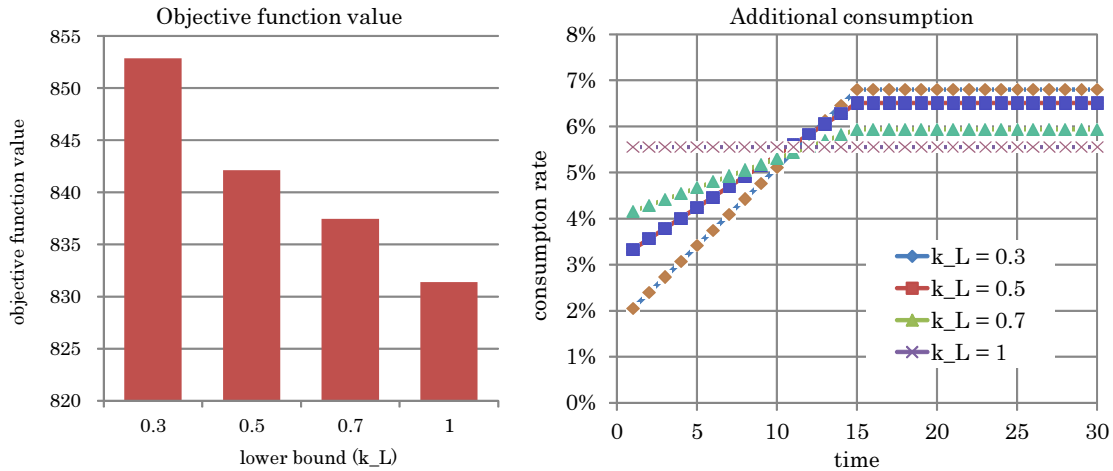

Figure 7: Lower bound of additional consumption

The result also shows the multiple of $C_{\alpha,1}$ is equal to the lower bound as well. The objective function value increases as the lower bound becomes smaller. We have the trade-off relationship between the additional consumption in early period and the longevity risk. We find the lower bound is one of the important parameters to affect the result, while this shows the model can reflect the trade-off for different household needs. In this paper, we assume that the household wants to spend money in the early period as well as it wants to control the longevity risk, and we also set $k_L = 1$ hereafter.

We compare asset allocations between with and without private pension in Figure 8. The asset allocation at time 0 is shown in the left-hand side. We can purchase the private pensions at time 0. The fractions of the private pensions to the amount of initial wealth are 12% for the householder, and 42% for the spouse. More than 50% of wealth is spent to purchase the private pensions so that the household can obtain the stable incomes in the future. The amount of private pension of the spouse is much larger than that of the householder because the survival rate of the spouse is much higher than that of the householder,[6] and therefore the household needs to avoid the longevity risk for the spouse. In the case that we can purchase the private pension, we invest in more amount of a risky asset than the case without private pension because we can stabilize the future incomes by private pension. As the result, we can aim to increase the higher return by a risky asset as well as we avoid the shortfall risk by private pension. On the other hand, we invest the most amounts of wealth in cash in order to avoid shortfall risk when we do not join the private pension plan.

We show the asset allocation at time 1 and time 9 in the middle and right-hand side of Figure 8, respectively. We can make the state-dependent investment using the hybrid model, and find the relationship between the amount of wealth and weights. Regardless of the private pension, the weight of stock decreases in proportion to the amount of wealth. We

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[6]The survival rate of the spouse is usually higher than that of the householder. Moreover, the difference is larger because the subjective health feeling of the householder is 'good', whereas that of the spouse is 'excellent'.
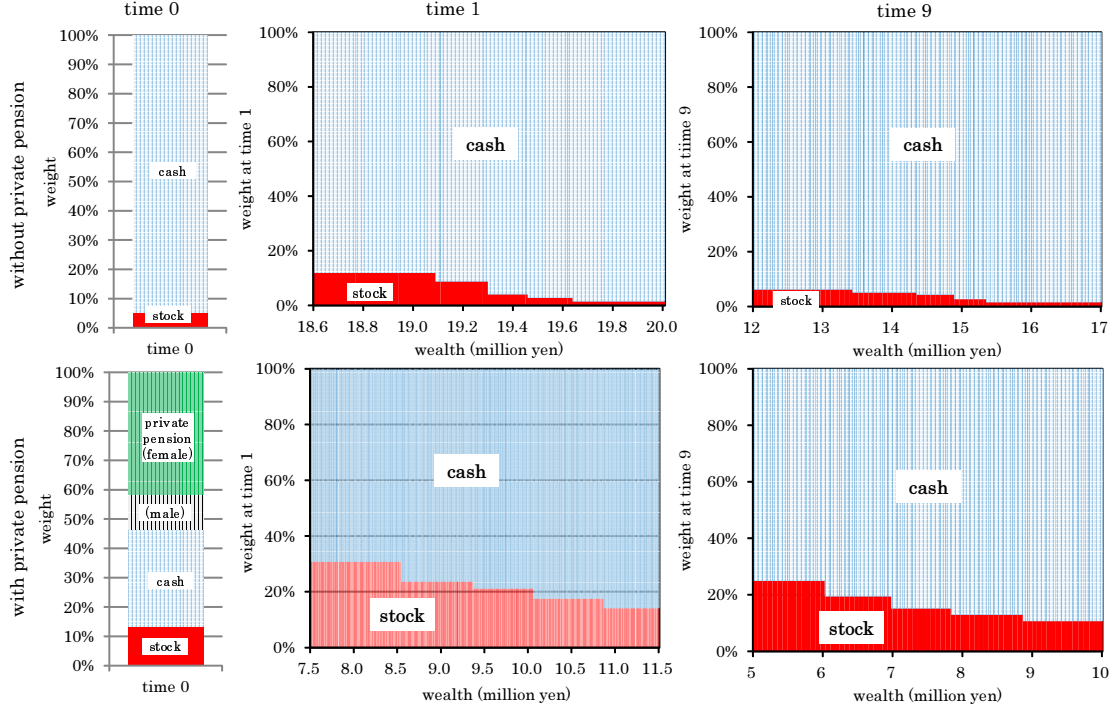
Figure 8: Asset allocation at time 0, 1, and 9

invest in more risky asset when the amount of wealth is small because we need to increase the amount of wealth and reduce longevity risk by taking investment risk.

4.2. Sensitivity of mortality rate

We test six combinations of subjective health feeling of family in Table 4.

Table 4: Combinations of subjective health feeling of family

	exex	expo	goex	gopo	poex	popo
householder(male)	excellent	excellent	good	good	poor	poor
spouse(female)	excellent	poor	excellent	poor	excellent	poor

We compare the objective function values on the left-hand side of Figure 9, and asset allocations at time 0 on the right-hand side.

The worse the subjective health feeling of householder is, the smaller the objective function value is, whereas the worse the feeling of spouse is, the larger the value is. The householder who has 'excellent' subjective health feeling can obtain more public pension in a whole retirement period than the 'poor' feeling, whereas the spouse who has 'poor' feeling exposes less longevity risk than the 'excellent' feeling. Therefore, the objective function value in the 'expo' case is the best, whereas the value in the 'poex' case is the worst.

The subjective health feeling affects the amount of pension. The worse the subjective health feeling is, the smaller the amount of pension is in the right-hand side of Figure 9. We compare the amount of pension between male and female. More amounts of pension are purchased when the feeling is 'excellent'. The reason is that a person who can expect to live longer needs to purchase a life annuity to avoid longevity risk. The sum of pensions of 'poex'(male with 'poor' feeling and female with 'excellent' feeling) is larger than those of 'expo'(male with 'excellent' feeling and female with 'poor' feeling) because the survival rate
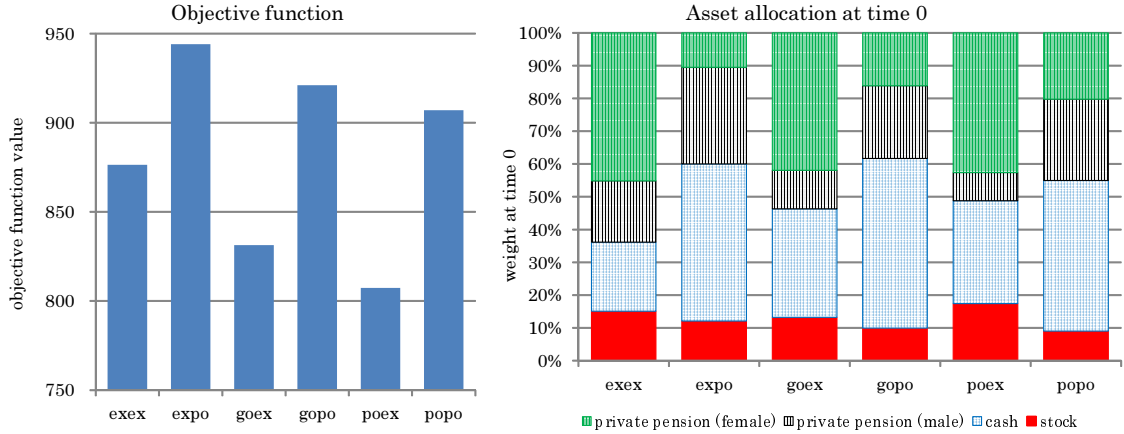
Figure 9: Objective function and asset allocation at time 0

of female with 'excellent' feeling is larger than that of male with 'excellent' as in Figure 3. When the subjective health feeling of female is 'poor', we have cash in place of pensions.

We show the additional consumption rate in Table 5. As in Section 4.1, the optimal rate of additional consumption to wealth is dependent on the subjective health feeling, but it is constant over time, regardless of subjective health feeling.

Table 5: Additional consumption rate

exex	expo	goex	gopo	poex	popo
5.53%	6.21%	5.55%	6.21%	5.60%	6.47%

The consumption rate for 'poor' feeling is larger than the rate for 'excellent' feeling because we can reduce the shortfall risk. We have a little difference among three kinds of subjective health feelings of male, whereas we have a large difference between 'poor' and 'excellent' feelings of female. The reason is we have a larger difference between survival rates of female than those of male as in Figure 2.

The survival rate affects the optimal allocation for a household, and therefore we find it is very important to propose the optimal retirement plan in consideration of the increase in life expectancy and the individual health status.

4.3. Sensitivity of inflation rate

We assume the inflation rate is 0% in Section 4.1. However, the inflation risk is critical for retirement planning, and therefore we conduct the sensitivity analysis for the different inflation rates; $f = 0\%, 0.1\%, 0.2\%, 0.3\%$.

At first, we show the average deficits over time in the left-hand side of Figure 10. The deficit shown in the graph is defined as the public pension minus minimum living expense and medical expenses. It does not include the income from private pension and the expenditure from the additional consumption computed by the optimization.

When the inflation occurs, the nominal minimum living expense and medical expenses increase, but the income from public pension remains fixed in spite of the macro-economic indexation because the inflation rate is smaller than the indexation adjustment rate which is 0.9%. Therefore the average deficits increase over time, and the longevity risk increases by the inflation.[7]

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[7]The average deficits reduce at time 5 and at time 10 because the self-pay ratio of medical expenses is
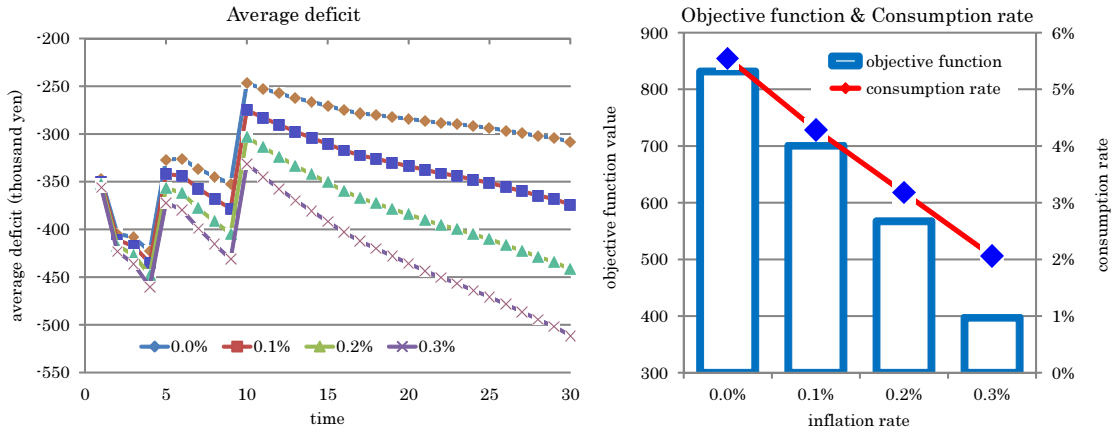
Figure 10: Average deficits, objective function and consumption rate

Next, we show the objective function value and the additional consumption rate in the right-hand side of Figure 10. The consumption rate is time-constant as well as the previous results. As the inflation rate increases, the objective function value decreases because the deficits increase and the additional consumption should be reduced. Therefore, they are highly correlated.

We show the asset allocation at time 0 and the average stock weight over time in Figure 11.
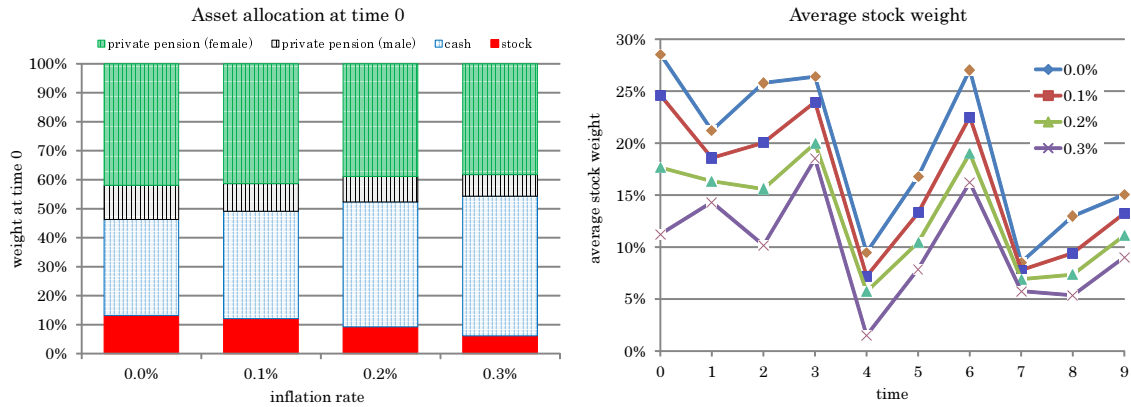

Figure 11: Asset allocation at time 0 and average stock weight

The amounts of private pensions decrease a little bit as the inflation rate rises because the income from the private pension remains fixed, and the shortfall risk increases. The cash ratio rises in proportion to the inflation rate because the deficits increase in the later planning period and we need to reduce the shortfall risk in the early planning period. When the inflation rate rises in the working generation, the wage income increases as well as the expenditures, and therefore investing in stock is a good solution in order to avoid the decrease in the monetary value. The inflation risk affects both the working generation and retirement, but the optimal investment strategies are opposite each other.

30% under 70 years old, but it decreases 20% between 70 and 74 years old, and 10% over 75 years old.

4.4. Impact of changing the public pension system

The income from public pension is essential for the retired household. But the government has the possibility to change the public pension system because it is in a tough situation. In this section, we assume the alternative to the current macro-economic indexation.[8] We assume the modified indexation where the adjustment rate is the inflation rate minus the indexation adjustment rate (0.9%) for any inflation rate.[9]

At first, we show the pension benefit over time on the left-hand side of Figure 12.
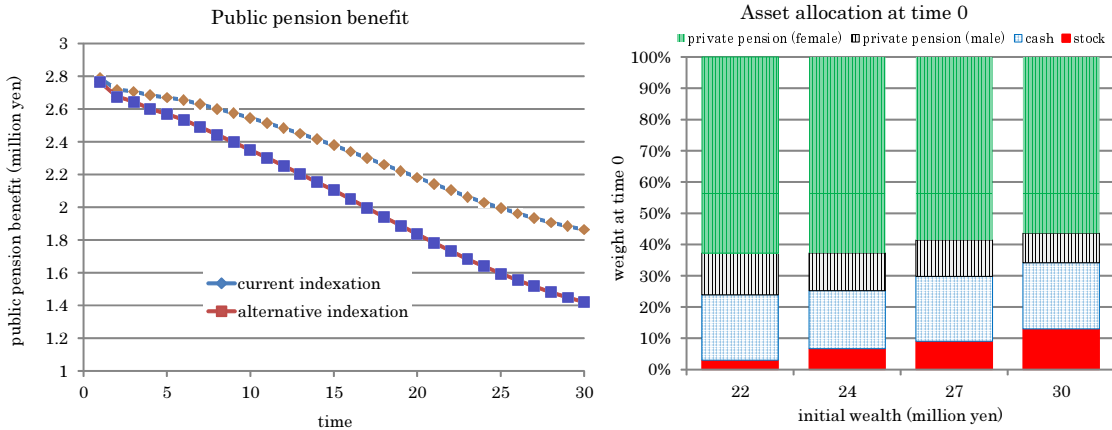

Figure 12: Public pension benefit and asset allocation

When the alternative indexation is adopted in place of the current indexation, the pension benefit reduces. As the results, the deficits also increase, and the amount of wealth decrease. When we solve the optimization problem under the alternative indexation with the base parameter of the initial wealth which is 20 million yen, the problem is infeasible and we cannot derive the solution. This means the household should prepare more amount of wealth until 65 years old, and the change of the public pension system impacts the household in a big way.

Under the alternative indexation, we conduct the sensitivity analysis for different initial amounts of wealth; $W_0 = 22, 24, 27, 30$ (million yen). We set the initial wealth so that we can derive the optimal solution.

We show the asset allocation at time 0 on the right-hand side of Figure 12. We purchase private pension under the alternative indexation more than current indexation because we need to increase the income from the private pension in place of the public pension. As the initial amount of wealth increases, the weight of private pension decreases, but we invest in stock and aim to get higher return. We need to prepare more amount of wealth until 65 years old in order to avoid changing risk of the public pension system. Therefore, life planning in the working generation is also important to build wealth as well as retirement planning.

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[8]The adjustment rate is the maximum of the inflation rate minus the indexation adjustment rate (0.9%) and zero under the current macro-economic indexation, as shown in Section 2.3.1. There are other alternatives, which are a rise in pension eligibility age and the increase in pension premium.

[9]This alternative is discussed in the Committee for Pension, the Social Security Council, Ministry of Health, Labour and Welfare[22].

5. Concluding remarks

In this paper, we propose the multi-period optimization model for retirement planning in the simulated path approach. We can provide investment strategies with the step function and consumption strategies, which are associated with the amount of wealth. Moreover, we determine the amount of life annuity to control longevity risk. We conduct the analysis for a hypothetical household. We show the results for different cases; effect of purchasing the private pension, sensitivities for mortality rate and inflation rate, and impact of changing the public pension system. We examine the usefulness of the model through detailed analyses.

In the future research, we need to construct the integrated model with life planning model in the working generation.

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