A New Portfolio Optimization Model in a Downside Risk Framework

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Abstract

The well-known mean-variance model and the downside risk model are used to investment decision problems for portfolio selection. The downside risk model uses the risk measure that focuses on return dispersions below a specific target, In this paper, an alternative downside risk model to the Mean/Lower Partial Moment model(MLPM model) proposed by Bawa *et al.* is proposed. The new downside risk model is called the Mean/Open-L Deviation model(MOLD model). While the MOLD model is similar to the MLPM model formulated as a nonlinear programming model, the following characteristics are mainly introduced. (1) The MOLD model is formulated as a linear programming model. (2) The risk measure 'OLD' is understandable, which is the weighted sum of the average shortfall below a specific target and the maximum shortfall. In addition, the relationship between the risk parameter of the MOLD model and that of the MLPM model is designated. The MOLD model proposed in this paper is characterized and is compared with the MLPM model using historical data of Tokyo Stock Exchange. The results show that the characteristics which do not exist in the MLPM model can be found in MOLD model, whereas portfolios selected using the MOLD model are similar to those of the MLPM model.

Keywords : Portfolio Selection; Optimization Model; Downside Risk; MOLD model; MLPM model

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1 Introduction

There are many mathematical programming approaches to the portfolio selection problems; the mean-variance model by Markowitz [10], the mean-absolute deviation(MAD) model by Konno [8], the MAD-skewness model by Konno and Yamazaki [9], the safety first model, and the downside risk model. The downside risk approach uses an intuitive measure of risk. The return dispersions below a specified target, or the target shortfall is defined as the downside risk measure. Bawa and Lindenberg [2] show the Mean-Lower Partial Moments (MLPM) model. The risk measure, called the lower partial moments (LPM), is described as;

$$LPM_k(r_G) = \int_{-\infty}^{r_G} (r_G - \tilde{r})^k f(\tilde{r}) d\tilde{r}, \tag{1}$$

where r_G is the target return, $\tilde{r}$ is the stochastic variable of return, and $f(\tilde{r})$ is the probability density function. The type of moment, k, specified in Equation (1) captures an investor's preference in terms of the downside risk. Given historical data or scenarios, $LPM_k(r_G)$ is rewritten as;

$$LPM_k(r_G) = \frac{1}{T}\sum_{t=1}^{T}\Big\{\max(r_G - r_t, 0)\Big\}^k, \tag{2}$$

where t is the time or the state, and r_t is the return at time(state) t. Let n be the number of assets, x_j be the proportion invested in the jth asset, $r_{j,t}$ be the tth return of the jth asset, and r_E be the expected return required by the investor. The portfolio selection problem in the MLPM framework is formulated as in **Program 1**.

【 Program 1 】

$$\textbf{minimize} \quad LPM_k(r_G) = \frac{1}{T}\sum_{t=1}^{T}\Big\{\max(r_G - r_t, 0)\Big\}^k \tag{3}$$

$$\textbf{subject to} \quad r_t = \sum_{j=1}^{n} r_{j,t} \cdot x_j, \quad (t = 1, \dots, T) \tag{4}$$

$$\frac{1}{T}\sum_{t=1}^{T} r_t \geq r_E \tag{5}$$

$$\sum_{j=1}^{n} x_j = 1 \tag{6}$$

$$x_j \geq 0, \quad (j = 1, \dots, n) \tag{7}$$

Figure 1 depicts the indifference curves of $LPM_k(r_G; r_1, r_2)$ for $T = 2$. The return on the portfolio r_1 at time 1 is on the horizontal axis, and the return r_2 at time 2 is on the vertical axis.

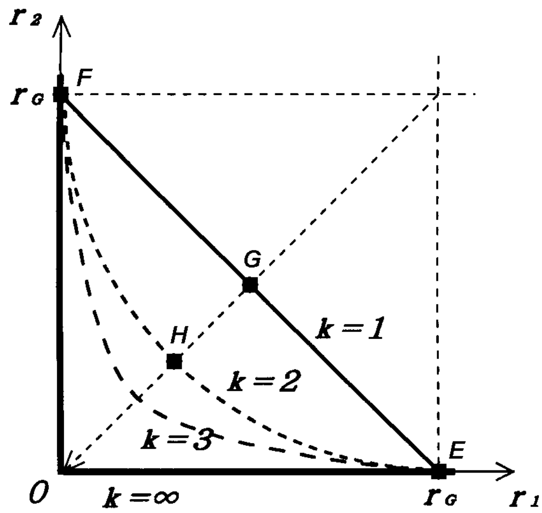

Figure 1: The indifference curves of $LPM_k(r_G; r_1, r_2)$ for $T = 2$

In the case of $k = 1$,

$$LPM_1(r_G; r_G, 0) = LPM_1(r_G; 0, r_G) = LPM_1(r_G; \frac{1}{2}r_G, \frac{1}{2}r_G).$$

These pairs of (r_1, r_2) on the line, such as point E, point F, and point G, are indifferent. In the case of $k = 2$,

$$LPM_2(r_G; r_G, 0) = LPM_2(r_G; 0, r_G) = LPM_2(r_G; \left(1 - \frac{1}{\sqrt{2}}\right) r_G, \left(1 - \frac{1}{\sqrt{2}}\right) r_G).$$

These pairs of (r_1, r_2) on the curve, such as point E, point F, and point H, are indifferent.

The MLPM model of $k = 1$ is formulated as a linear programming problem, that of $k = 2$ is formulated as a quadratic programming problem, and that of $k \geq 3$ is formulated as a nonlinear programming problem. The MLPM model of $k = \infty$ is formulated as a linear programming model, because the indifference curve of $k = \infty$ is the L-shaped curve, or a piecewise lenear curve.

In this paper, we propose an alternative downside risk model to the MLPM model, and call it the Mean–Open-L Deviation (MOLD) model. The MOLD model has the mixed objective function of $k = 1$ of the MLPM model and $k = \infty$, and substitutes for the nonlinear objective function of the MLPM model when $k \geq 2$. Since the objective function of $k = 1$ is mixed with the linear objective function of $k = \infty$, the MOLD model is formulated as a linear programming problem. While the MOLD model is similar to the MLPM model formulated as a nonlinear programming model, the following characteristics are mainly introduced.

(1) The MOLD model is formulated as a linear programming model.

(2) The risk measure 'OLD' is understandable, which is the weighted sum of the average shortfall below a specific target and the maximum shortfall.

(3) The risk parameter λ is introduced in order to designate the investor's preference with respect to the downside risk.

The paper is organized as follows. Section 2 introduces the formulation of the MOLD model. Section 3 discusses the relationship between the risk parameter λ of the MOLD model and k of the MLPM model. Sections 4 and 5 present numerical tests using historical data of Tokyo Stock Exchange. The MOLD model is analyzed and is compared with the MLPM model. Section 6 provides some concluding remarks and our future research.

2 The Mean–Open-L Deviation (MOLD) model

Figure 2 also depicts the indifference curves of $LPM_k(r_G; r_1, r_2)$ and open-L shaped curve for $T = 2$, as in *Figure* 1. We describe the relationship between the MLPM model and the MOLD model.

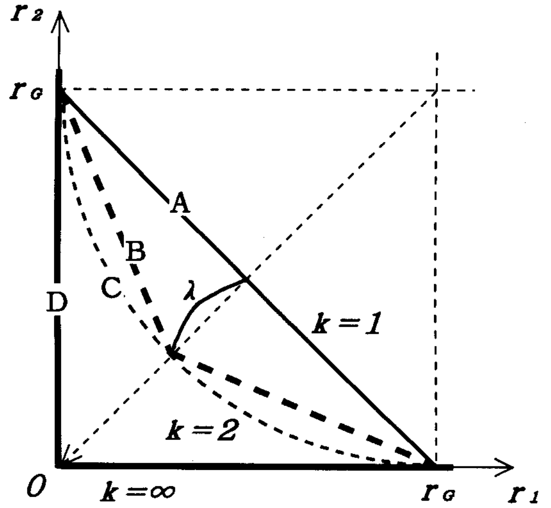

Figure 2: The indifference curves of $LPM_k(r_G)$ and the Open-L shaped curve

The line A($k = 1$) and the L-shaped curve D($k = \infty$) combine into the piecewise linear curve(B) in order to substitute for the indifference curve C($k = 2$) of $LPM_k(r_G; r_1, r_2)$. We call this piecewise linear curve the open-L shaped curve [2] . The piecewise linear curve cannot exactly fit the indifference curve of the MLPM model. However, the investor's preference of the MOLD model can substitute for that of the MLPM model by changing the risk parameter λ, corresponding to the risk parameter k. We call this alternative model the MOLD model. The MOLD model is formulated as in **Program 2** :

【 Program 2 】

$$\textbf{minimize} \quad OLD_\lambda(r_G) = (1-\lambda_k) \cdot \left\{ \frac{1}{T} \sum_{t=1}^{T} \max(r_G - r_t, 0) \right\} + \lambda_k \cdot \max_t \{\max(r_G - r_t, 0)\ ;\ t = 1, \ldots, T\} \tag{8}$$

subject to Equations(4) through (7)

[2]The objective function for the open-L shaped model of goal vector approach is the open-L shaped function [4].

where $(1-\lambda_k)$ is the proportion of LPM_1, and λ_k is the proportion of LPM_∞. The parameter λ_k provides the investor's risk preference. $\lambda_k = 0$ of the MOLD model corresponds to $k = 1$ of the MLPM model, and $\lambda_k = 1$ of the MOLD model corresponds to $k = \infty$ of the MLPM model. The risk measure is defined as $OLD_\lambda(r_G)$. **Program 2** is rewritten to the linear programing problem (**Program 3**) :

【 **Program 3** 】

minimize $$OLD_\lambda = (1-\lambda_k) \cdot \left(\frac{1}{T} \sum_{t=1}^{T} d_t^- \right) + \lambda_k \cdot d \tag{9}$$

subject to $$\sum_{j=1}^{n} (r_{j,t} - r_G) \cdot x_j + d_t^- \geq 0, \quad (t = 1, \ldots, T) \tag{10}$$

$$d_t^- - d \leq 0, \quad (t = 1, \ldots, T) \tag{11}$$

$$\sum_{j=1}^{n} R_j \cdot x_j \geq r_E \tag{12}$$

$$\sum_{j=1}^{n} x_j = 1 \tag{13}$$

$$x_j \geq 0, \quad (j = 1, \ldots, n) \tag{14}$$

$$d_t^- \geq 0, \quad (t = 1, \ldots, T) \tag{15}$$

$$d \geq 0 \tag{16}$$

where $R_j = \frac{1}{T} \sum_{t=1}^{T} r_{j,t}$, and $r_t = \sum_{j=1}^{n} r_{j,t} \cdot x_j$. $d_t^- = \max(r_G - r_t, 0)$, and $d = \max_t \ d_t^-$ result from **Program 3**.

3 Modelling the relationship between the risk parameters λ and k

We explain the derivation of $\lambda(k, T)$ when the open-L shaped curve(B) of the MOLD model substitutes for the indifference curve(C) of the MLPM model in *Figure* 2. *Figure* 3 depicts the derivation of the risk parameter λ of the MOLD model, corresponding to k of the MLPM model.

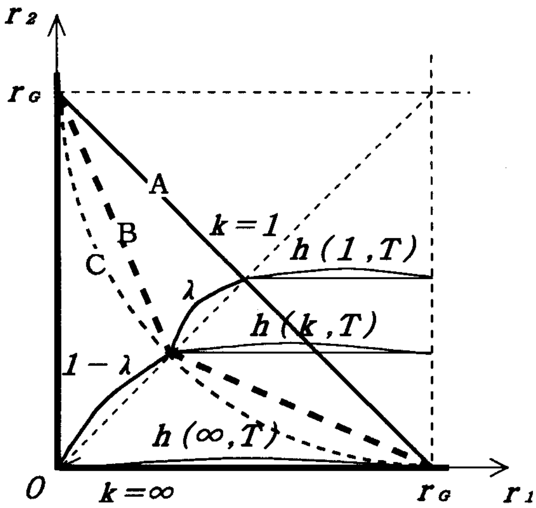

Figure 3: The derivation of the risk parameter λ of the MOLD model, corresponding to k of the MLPM model

We calculate LPM_k on the two points of the curve C, as follows;

(1) When only one of the returns is equal to 0, and the others are equal to r_G, $LPM_k(r_G) = \frac{1}{T} \cdot (r_G)^k$.

(2) When $r_t = r_G - h(k, T)$ for all t, $LPM_k(r_G) = \{h(k, T)\}^k$.

Since two points on the curve C are indifferent, $\{h(k, T)\}^k = \frac{1}{T} \cdot (r_G)^k$. Thus $h(k, T)$ in *Figure* 3 can be calculated as in Equation (17).

$$h(k, T) = \sqrt[k]{\frac{1}{T}} \cdot r_G \tag{17}$$

Since $\lambda(k, T)$ can be calculated using $h(k, T)$ as in Equation (18), Equation (19) can be derived.

$$\begin{aligned} \lambda(k, T) &= \frac{h(k, T) - h(1, T)}{r_G - h(1, T)} &(18) \\ &= \frac{1}{T-1} \cdot \left(\frac{T}{\sqrt[k]{T}} - 1 \right) &(19) \end{aligned}$$

Figure 4 depicts the relationship between λ and k for the various number of data T.

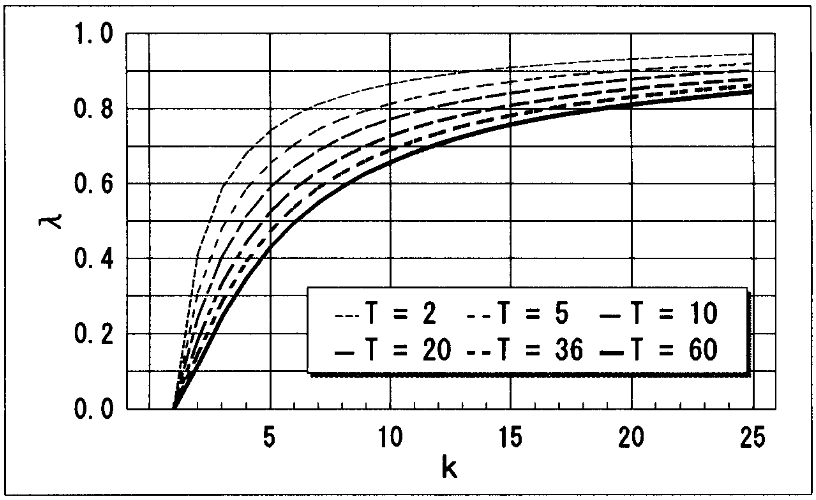

Figure 4: The relationship between λ and k

If k is constant, the larger T is, the smaller λ is. We need to evaluate the maximum value of d_t^-, depending on the value of T.

4 Numerical tests for the MOLD model

4.1 Data set and the parameters

We test the MOLD model using historical data of Tokyo Stock Exchange. Monthly data of stock returns are collected for four three-years listed in *Table* 1.

Table 1: Data period and the number of stocks

	Data period (36 months)			the number of stocks
Period A	January 1988	–	December 1990	1,078
Period B	January 1989	–	December 1991	1,109
Period C	January 1990	–	December 1992	1,140
Period D	January 1991	–	December 1993	1,173

We choose some parameters : three kinds of target returns ($r_G = 0.0\%, 0.2\%, 0.4\%$), two kinds of required expected returns ($r_E = 0.0\%, 0.5\%$), and two kinds of the upper limit invested in each stock ($U_j = 100\%$(NUL: No Upper Limit), 5%). The twelve combination of the parameters are analyzed. All of the problems are solved with XPRESS-MP(LP solver) for the MOLD model.

4.2 The relationship between the risk parameter λ and the indices of returns

We change the risk parameter λ parametrically at intervals of 0.01 from 0 through 1. 101 kinds of λ are tested.

First, we show six kinds of statistics in *Figure* 5 : mean, standard deviation, skewness, kurtosis, maximum, and minimum. Each statistics is calculated at each upper limit (NUL, $U_j = 5\%$), and it is the average of 24 statistics, or six kinds of parameters ($r_G \times r_E$) multiplied by four periods.

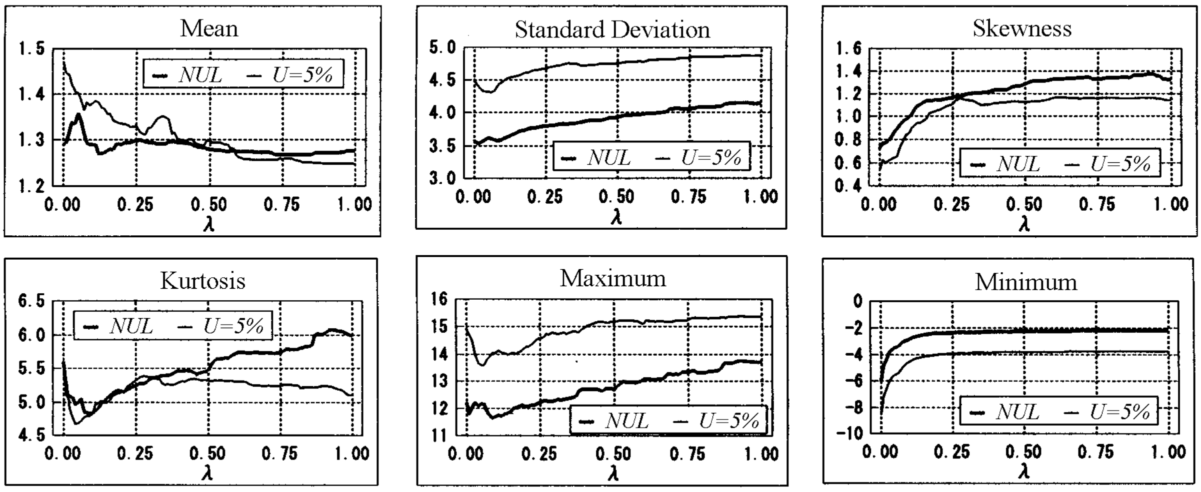

Figure 5: Statistics of returns

The minimum return is large, or the maximum target shortfall is small when the risk parameter λ is large. This is the reason that the maximum target shortfall affects the risk measure (OLD) largely when λ is large. The minimum returns are explicitly large when $\lambda = 0$ through $\lambda = 0.1$. However, they are saturated when $\lambda \geq 0.1$. The skewness tends to be large when λ is large. The minimum return affects the skewness. Whether the upper limit is included or not affects the results.

Next, we show the other indices of returns in *Figure* 6 [3] : OLD, number of invested stocks, maximum proportion, average target shortfall, number of periods to be maximum target shortfall (# of periods[max. shortfall]), and number of periods below the target (# of periods[shortfall]).

[3] Indices are defined as follows;

- number of invested stocks = $\{\text{number of } j | x_j > 0\}$
- maximum proportion = $\max_j \; x_j$
- average target shortfall = $\frac{1}{T}\sum_{t=1}^{T} d_t^-$
- # of periods[max. shortfall] = $\{\text{number of } t | d_t^- = \max_s d_s^-\}$
- # of periods[shortfall] = $\{\text{number of } t | d_t^- > 0\}$

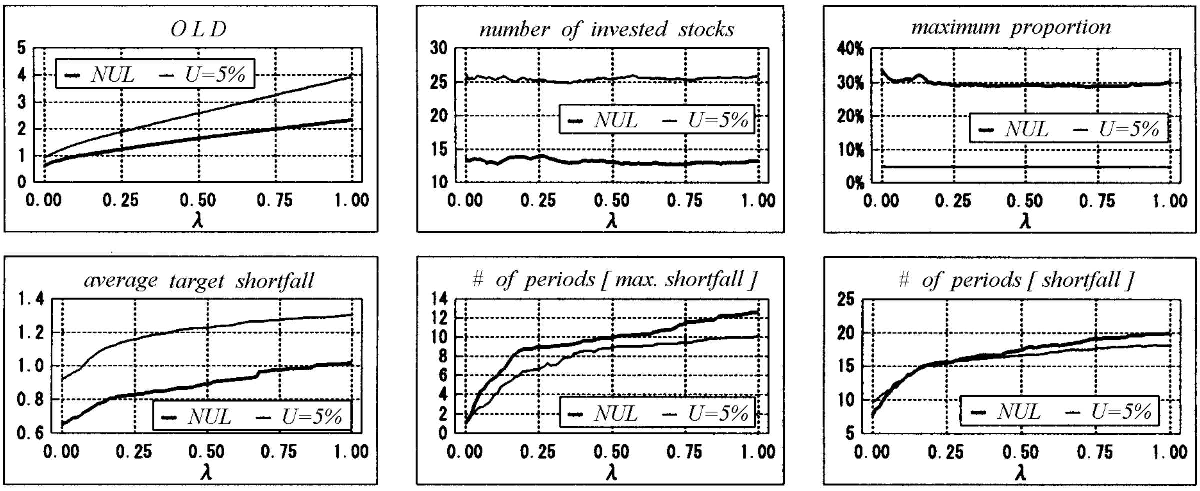

Figure 6: The other indices of returns

The average target shortfall is large (small) while the minimum return is large (small), that is, the maximum target shortfall is small (large). This reason is that both indices are included in the objective function, and these have the trade-off relationship. The users of the MOLD model should notice this relationship. We can find the following information. When the risk parameter λ is large,

- the minimum return is large, or the maximun target shortfall is small,
- the average target shortfall is large,
- the standard deviation and the skewness tend to be large,
- the number of the periods of the maximum target shortfall tends to increase,
- the number of the periods below the target, or the probability below the target, tends to increase.

We cannot characterize the number of invested stocks and the maximum proportion.

4.3 The relationship between the risk parameter λ and the proportion invested in the stocks

The relationship between λ and the proportion is discussed in the period D (January 1991 – December 1993). *Table* 2 shows the average of 101 kinds of proportions derived by solving the problems with λ at intervals of 0.01 from 0 through 1. The numbers in the first (left) column are the security codes of invested stocks.

Table 2: The average of the proportions

UL	NUL	NUL	NUL	NUL	NUL	NUL	5%	5%	5%	5%	5%	5%	NUL	5%	
$r_E =$	0.5	0.5	0.5	1.0	1.0	1.0	0.5	0.5	0.5	1.0	1.0	1.0			total
$r_G =$	0.0	0.2	0.4	0.0	0.2	0.4	0.0	0.2	0.4	0.0	0.2	0.4	average	average	average
8529	19.74%	18.87%	18.31%	11.54%	11.20%	10.66%	5.00%	5.00%	5.00%	4.89%	4.90%	4.90%	15.051%	4.949%	10.000%
7269	9.32%	9.13%	9.60%	12.07%	12.24%	12.69%	5.00%	5.00%	5.00%	5.00%	5.00%	5.00%	10.841%	5.000%	7.921%
6332	5.19%	5.17%	4.70%	13.75%	13.76%	13.00%	4.73%	4.69%	4.68%	5.00%	5.00%	5.00%	9.262%	4.849%	7.055%
8182	9.24%	9.29%	9.39%	8.73%	8.90%	8.98%	5.00%	5.00%	5.00%	4.70%	4.65%	4.55%	9.087%	4.817%	6.952%
8273	6.58%	6.93%	7.54%	8.46%	8.85%	9.59%	4.99%	4.93%	4.75%	5.00%	5.00%	5.00%	7.990%	4.943%	6.467%
8196	6.60%	6.13%	5.55%	10.57%	9.86%	9.40%	4.73%	4.67%	4.62%	4.46%	4.49%	4.60%	8.016%	4.592%	6.304%
9508	7.44%	7.15%	6.69%	7.93%	7.83%	7.29%	5.00%	5.00%	5.00%	5.00%	5.00%	5.00%	7.387%	5.000%	6.194%
2810	1.16%	1.97%	3.49%	5.51%	6.10%	7.72%	5.00%	5.00%	5.00%	5.00%	5.00%	4.99%	4.325%	4.998%	4.662%
6783	5.51%	5.42%	5.36%	4.17%	4.11%	4.05%	4.89%	4.93%	4.97%	3.87%	3.80%	3.96%	4.769%	4.403%	4.586%
8536	9.71%	9.98%	9.82%	0.22%	0.22%	0.40%	5.00%	5.00%	5.00%	0.52%	0.62%	1.15%	5.059%	2.880%	3.969%
8530	4.94%	4.89%	5.24%	0.13%	0.23%	0.30%	5.00%	5.00%	5.00%	4.02%	4.01%	3.77%	2.622%	4.466%	3.544%
8551	3.69%	4.20%	3.91%	0.00%	0.01%	0.00%	5.00%	5.00%	5.00%	5.00%	5.00%	5.00%	1.969%	5.000%	3.484%
2212	0.00%	0.08%	0.04%	3.64%	4.13%	4.35%	4.52%	4.45%	4.32%	5.00%	5.00%	5.00%	2.040%	4.715%	3.377%
8264	1.41%	1.68%	1.78%	1.68%	1.51%	1.22%	3.80%	3.28%	2.78%	4.79%	4.45%	4.00%	1.546%	3.850%	2.698%
9507	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	3.88%	3.85%	3.95%	5.00%	5.00%	5.00%	0.000%	4.445%	2.222%
9665	0.01%	0.00%	0.00%	2.81%	2.50%	1.90%	0.71%	0.72%	0.75%	4.75%	4.64%	4.52%	1.202%	2.681%	1.941%
5110	0.36%	0.45%	0.51%	0.29%	0.24%	0.19%	3.06%	2.70%	2.27%	3.97%	3.84%	3.54%	0.339%	3.230%	1.785%
8395	1.40%	1.23%	1.56%	4.12%	4.46%	4.56%	0.10%	0.10%	0.06%	0.21%	0.13%	0.23%	2.887%	0.139%	1.513%
8350	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	3.20%	3.28%	3.47%	2.24%	2.35%	2.21%	0.000%	2.790%	1.395%
8165	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	1.47%	1.71%	1.98%	3.03%	3.26%	3.43%	0.000%	2.480%	1.240%
8183	2.17%	1.56%	0.87%	2.14%	1.31%	0.35%	0.18%	0.40%	0.48%	1.85%	1.52%	1.55%	1.398%	0.997%	1.198%
9735	1.35%	1.87%	1.74%	0.00%	0.07%	0.24%	0.89%	1.07%	1.21%	1.29%	1.86%	1.92%	0.879%	1.372%	1.126%
1924	0.42%	0.30%	0.08%	0.01%	0.07%	0.08%	0.00%	0.00%	0.00%	3.76%	4.26%	4.21%	0.161%	2.039%	1.100%
4536	3.25%	2.89%	2.59%	0.07%	0.01%	0.07%	0.40%	0.44%	0.55%	0.79%	0.74%	0.76%	1.480%	0.614%	1.047%
8018	0.39%	0.42%	0.67%	0.11%	0.45%	0.71%	2.31%	3.19%	3.92%	0.00%	0.00%	0.19%	0.457%	1.602%	1.030%

The parameters r_E and r_G affect the invested proportions explicitly. *Table* 3 indicates the difference.

Table 3: The difference of the proportions

	SD_e		SD_g		
	$r_E = 0.5\%$	$r_E = 1.0\%$	$r_G = 0.0\%$	$r_G = 0.2\%$	$r_G = 0.4\%$
NUL	1.87%	1.98%	9.45%	9.49%	9.30%
$U = 5\%$	1.24%	0.85%	5.55%	5.60%	5.50%

SD_e indicates the difference of proportions due to the target return, and SD_g indicates the difference of proportions due to the required expected return. Let x_{jeg} denote the proportion where $j \in J = \{l | x_{leg} \neq 0\}$, $e \in \{0.5\%, 1.0\%\}$, and $g \in \{0.0\%, 0.2\%, 0.4\%\}$. SD_e and SD_g are calculated as follows :

$$SD_e = \frac{1}{2}\sqrt{\sum_{j \in J}(\max_g x_{jeg} - \min_g x_{jeg})^2} \tag{20}$$

$$SD_g = \frac{1}{2}\sqrt{\sum_{j \in J}(\max_e x_{jeg} - \min_e x_{jeg})^2} \tag{21}$$

Since all of SD_g are larger than SD_e, the required expected return r_E affects the proportions in comparison with the target return r_G.

Next, consider the common proportions from $\lambda = 0$ through $\lambda = \lambda'$, where λ' changes from $\lambda' = 0$ through $\lambda' = 1$. *Figure* 7 depicts the common proportions at each upper limit in the period D.

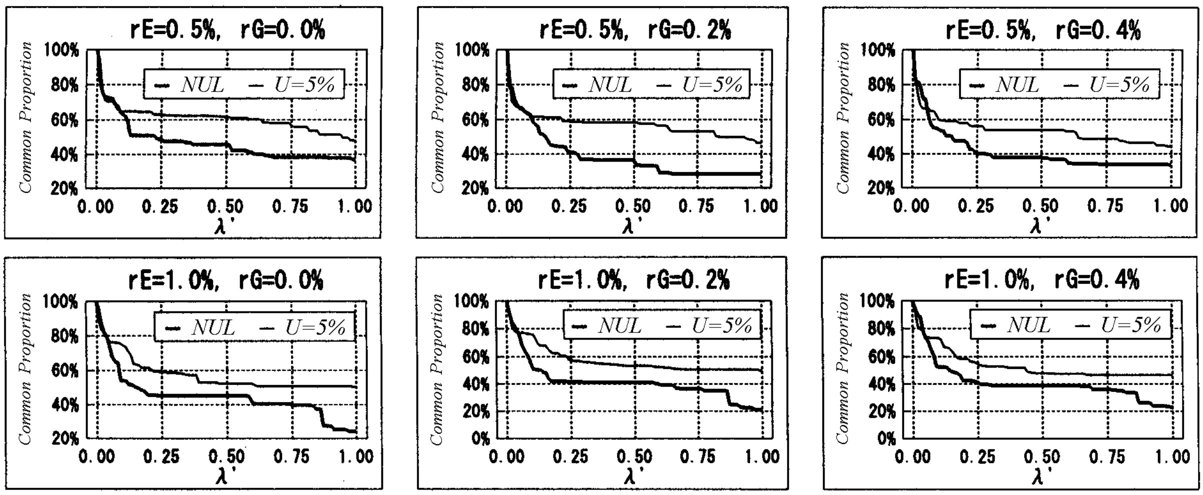

Figure 7: The common proportions in the period D ($\lambda = 0 \sim \lambda'$)

When λ' is large, the decrease of the common proportions is small, or proportions invested in the stocks do not change even if the risk parameter λ changes. In the range from $\lambda = 0$ through $\lambda = 0.1$, the proportions change drastically. The maximum target shortfall affects the proportions in the range from $\lambda = 0$ to $\lambda = 0.1$. The common proportions with the upper limit of 5% tend to be larger than those with no upper limit, and about 50% invested in the stocks with the upper limit of 5% is common regardless of λ.

Figure 8 depicts the common proportions from $\lambda = \lambda' - \alpha$ to $\lambda = \lambda' + \alpha$, where $\alpha = 0.1, 0.05, 0.01$. The period is D, and parameters are $r_E = 0.5\%$ and $r_G = 0.0\%$. λ' changes from $\lambda' = 0$ through $\lambda' = 1$.

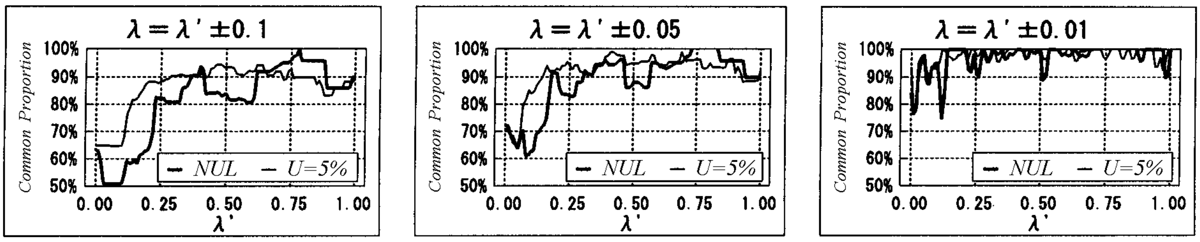

Figure 8: The common proportions from $\lambda = \lambda' - \alpha$ to $\lambda = \lambda' + \alpha$
(period D, $r_E = 0.5\%$, $r_G = 0.0\%$)

The larger λ' is, the larger the common proportion tends to be. The common proportion exceeds 80% in the case of $\lambda' \geq 0.25$ and $\alpha = 0.1$. We can derive the useful information about the sensitivity of proportions to the risk parameter λ.

Figure 9 depicts the top six proportions of stocks with no upper limit in the period D. The average proportions when using three kinds of target returns are shown.

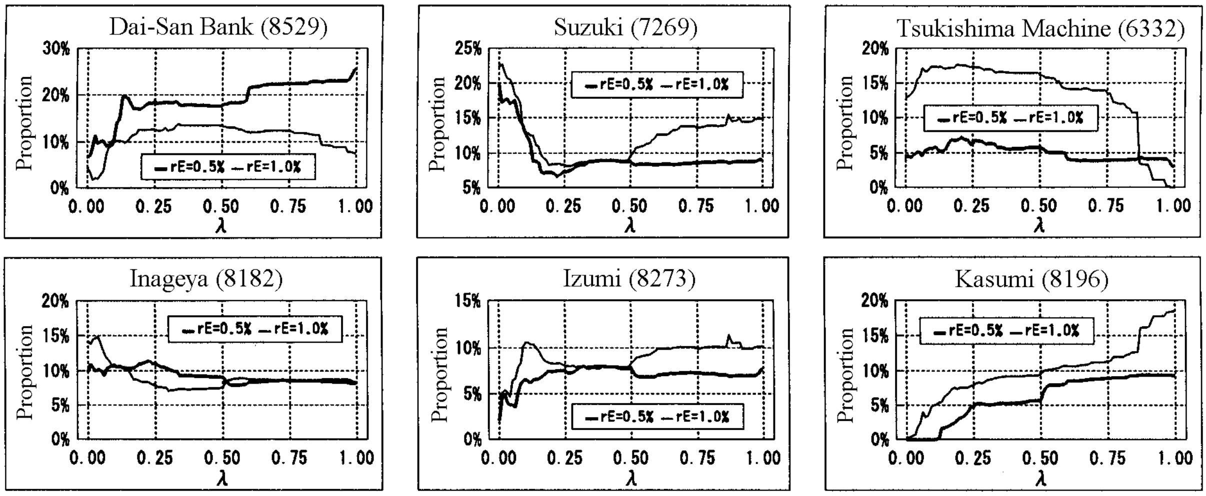

Figure 9: The top six proportions of stocks

The information in *Figure* 9 helps the investors select their portfolio effectively even if they cannot determine the risk parameter (preference) explicitly.

4.4 The efficient frontier

Figure 10 depicts the efficient frontiers in the expected return–OLD space of the MOLD model. There are the efficient frontiers to six kinds of λ, corresponding to $k = 1$ through $k = 5$, and $k = \infty$. Three curves from left to right show the frontiers in the case of $r_G =$ 0.0%, 0.2%, 0.4%, respectively.

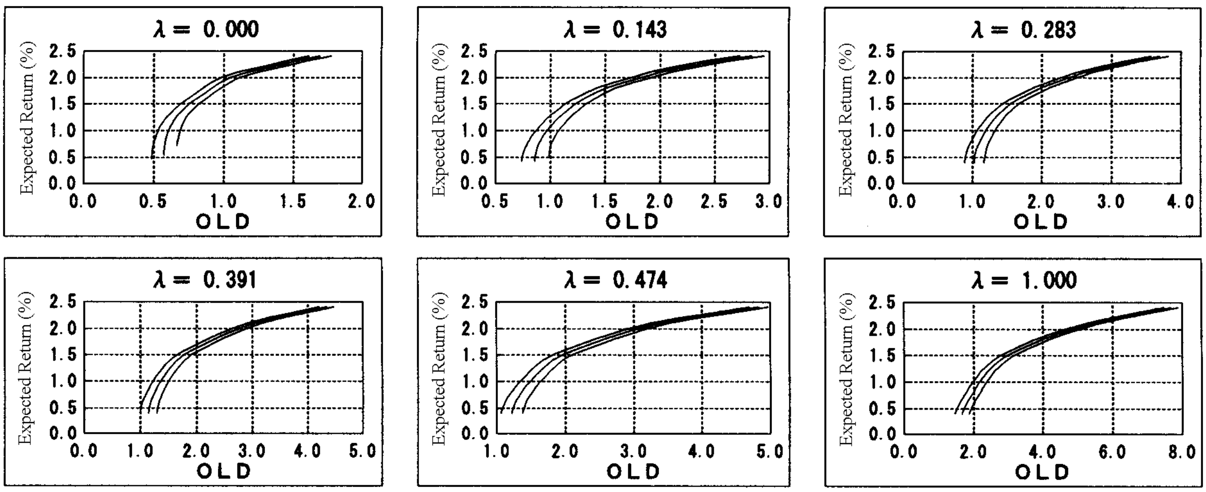

Figure 10: The efficient frontiers of the MOLD model (period D)

We can designate the trade-off relationship between return and risk, and make investment decisions using the efficient frontiers of the MOLD model.

5 The comparison between the MOLD model and the MLPM model

The MOLD model is compared with the MLPM model. We use the same data as in Section 4. The software NUOPT is used to solve the nonlinear programming problems (MLPM model).

5.1 The return distribution

Figure 11 depicts the frequency and the cumulative frequency of returns. Those are calculated using the optimal proportions. The period is D, $r_E = 0.5\%$, $r_G = 0.0\%$, and the upper limit is not constrained. The top six graphs are the histograms of the MOLD model, the middle six graphs are the histograms of the MLPM model, and the lower six graphs are the cumulative frequency distributions, where the solid lines are for the MOLD model and the dotted lines are for the MLPM model [4] 。

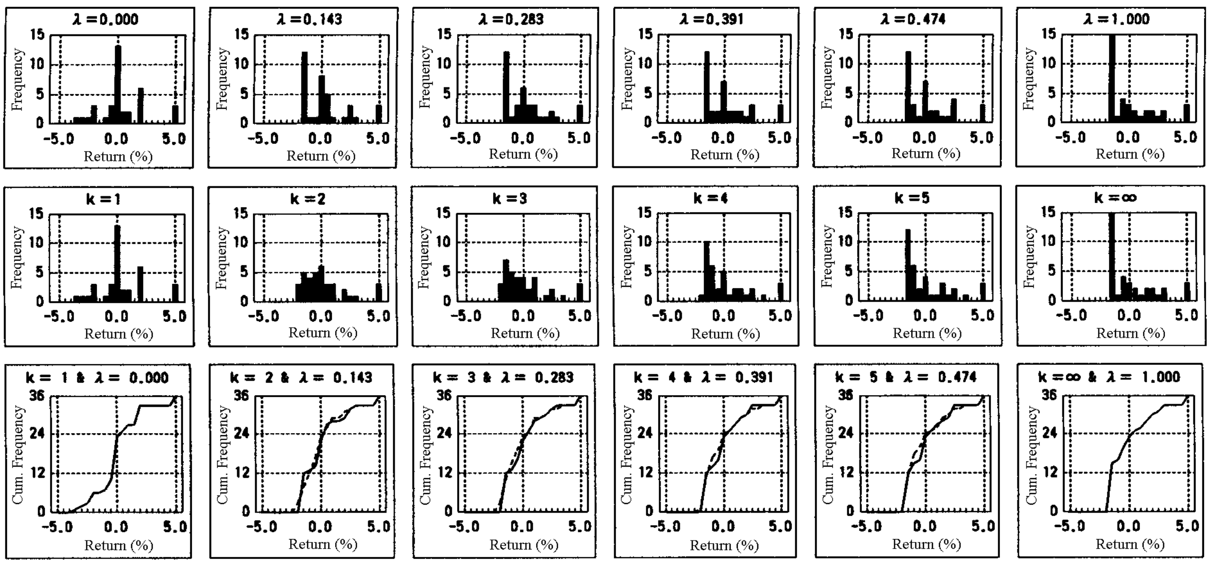

Figure 11: The return histograms and the return cumulative frequency distributions (period D, $r_E = 0.5\%$, $r_G = 0.0\%$, and no upper limit)

Histograms of both models look different, while the cumulative frequency distributions of both models look similar. The following results are obtained;

(1) the maximum target shortfall of the MOLD model is smaller than that of the MLPM model,

(2) the number of periods below the target of the MOLD model is smaller than that of MLPM model,

[4] More than 5% returns aggregate to 5% returns in the graphs.

(3) the average target shortfall of the MOLD model tends to be smaller than that of the MLPM model,

(4) the number of periods to be maximum target shortfall of the MOLD model is larger than that of the MLPM model.

5.2 The proportions invested in stocks

Table 4 shows the proportions of the same stocks both models have in common. The average proportions in six kinds of parameters are listed.

Table 4: The common proportions of the MOLD model and the MLPM model

	No Upper Limit					Upper Limit = 5%				
(MLPM) $k =$	2	3	4	5		2	3	4	5	
(MOLD) $\lambda =$	0.143	0.283	0.391	0.474	average	0.143	0.283	0.391	0.474	average
period A	96.34%	97.18%	96.25%	95.04%	96.20%	91.80%	88.06%	89.48%	89.04%	89.60%
period B	91.64%	93.88%	93.91%	93.85%	93.32%	84.46%	89.63%	89.50%	90.02%	88.40%
period C	84.89%	84.24%	84.98%	85.75%	84.96%	75.23%	78.17%	83.33%	88.41%	81.28%
period D	79.57%	79.07%	80.97%	79.42%	79.76%	83.57%	85.47%	88.99%	90.11%	87.04%
average	88.11%	88.59%	89.03%	88.52%	88.56%	83.76%	85.34%	87.82%	89.40%	86.58%

When k is large, both models tend to have the same stocks in common, or the common proportions of both models are large. The common proportions are more than 80% on average. Thus the MOLD model may alternate with the MLPM model when investors have the portfolio under the downside risk criterion.

Next, *Table* 5 shows the common proportions between two values in each risk parameter. The average proportions in six kinds of parameters in the period D are listed.

Table 5: The common proportions between two values in each risk parameter (period D)

NUL		$\lambda = 0.143$	$\lambda = 0.283$	$\lambda = 0.391$	$\lambda = 0.474$	MOLD
		58.86%	54.86%	55.90%	55.90%	$\lambda = 0.000$
$k = 2$	64.92%		82.07%	79.72%	79.68%	$\lambda = 0.143$
$k = 3$	58.28%	89.00%		91.89%	91.96%	$\lambda = 0.283$
$k = 4$	55.54%	84.90%	94.28%		98.74%	$\lambda = 0.391$
$k = 5$	54.87%	83.40%	89.99%	94.93%		
MLPM	$k = 1$	$k = 2$	$k = 3$	$k = 4$		

$U_j = 5\%$		$\lambda = 0.143$	$\lambda = 0.283$	$\lambda = 0.391$	$\lambda = 0.474$	MOLD
		66.82%	60.93%	58.97%	57.71%	$\lambda = 0.000$
$k = 2$	68.12%		88.25%	85.12%	83.66%	$\lambda = 0.143$
$k = 3$	64.28%	90.96%		94.72%	93.00%	$\lambda = 0.283$
$k = 4$	61.55%	85.63%	93.57%		97.12%	$\lambda = 0.391$
$k = 5$	58.10%	82.40%	89.93%	95.40%		
MLPM	$k = 1$	$k = 2$	$k = 3$	$k = 4$		

The right-upper triangles show the values of the MOLD model, and the left-lower triangles show the values of the MLPM model. For example, 82.07% where $\lambda = 0.143$ and $\lambda = 0.283$ cross with no upper limit (NUL) is the common proportion in both risk parameter values of the MOLD model. 89.00% where $k = 2$ and $k = 3$ cross with NUL is the common proportion in both risk parameters of the MLPM model. When $k \geq 3$, the common proportions are large. We can also compare the MOLD model with the MLPM model in *Table* 5. The abovementioned 82.07% of the MOLD model is compared with 89.00% of the MLPM model. Both models construct the similar portfolio. The similar results can be obtained in the other periods.

5.3 The LPM values

We compare both models using the LPM values, rather than using the return distributions and the invested proportions. We define $\sqrt[k]{LPM_k}$ as the LPM value of solving the MLPM problem, and define $\sqrt[k]{LPM(k,\lambda)}$ as the LPM value of solving the MOLD problem;

$$\sqrt[k]{LPM_k} = \sqrt[k]{\frac{1}{T}\sum_{t=1}^{T}\left(d_{t,MLPM}^{-*}\right)^k}, \tag{22}$$

$$\sqrt[k]{LPM(k,\lambda)} = \sqrt[k]{\frac{1}{T}\sum_{t=1}^{T}\left(d_{t,MOLD}^{-*}\right)^k}, \tag{23}$$

where $d_{t,MLPM}^{-*}$ is the target shortfall of solving the MLPM model, and $d_{t,MOLD}^{-*}$ is that of solving the MOLD model. Let $\min_{\lambda}\left[\sqrt[k]{LPM(k,\lambda)}\right]$ denote the minimum value of 101 kinds of $\sqrt[k]{LPM(k,\lambda)}$ of solving the MOLD model at intervals of 0.01 from $\lambda = 0$ through $\lambda = 1$. We evaluate the difference between LPM values of the MOLD model and those of the MLPM model;

$$DV_k = \frac{\sqrt[k]{LPM(k,\lambda_k)} - \sqrt[k]{LPM_k}}{\sqrt[k]{LPM_k}} = DV_k^{(1)} + DV_k^{(2)}, \tag{24}$$

$$DV_k^{(1)} = \frac{\min\limits_{\lambda}\left[\sqrt[k]{LPM(k,\lambda)}\right]}{\sqrt[k]{LPM_k}} - 1, \tag{25}$$

$$DV_k^{(2)} = \frac{\sqrt[k]{LPM(k,\lambda_k)} - \min\limits_{\lambda}\left[\sqrt[k]{LPM(k,\lambda)}\right]}{\sqrt[k]{LPM_k}}. \tag{26}$$

$DV_k^{(1)}$ is the difference rate caused by approximating the MLPM model to the linear model, and $DV_k^{(2)}$ is the difference rate caused by the relation model between λ and k, or Equation (19). *Table* 6 shows the values of $DV_k^{(1)}$ and $DV_k^{(2)}$. The average of six kinds of these values are listed.

Table 6: The difference rate of LPM values

		No Upper Limit					$U_j = 5\%$				
		$k=2$	$k=3$	$k=4$	$k=5$	average	$k=2$	$k=3$	$k=4$	$k=5$	average
period A	$DV_k^{(1)}$	5.39%	3.91%	3.41%	3.27%	4.00%	10.01%	8.36%	7.33%	5.73%	7.86%
	$DV_k^{(2)}$	0.72%	0.92%	1.69%	0.17%	0.88%	0.29%	1.69%	0.74%	0.88%	0.90%
	DV_k	6.11%	4.83%	5.11%	3.44%	4.87%	10.30%	10.05%	8.07%	6.60%	8.75%
period B	$DV_k^{(1)}$	4.45%	3.57%	3.42%	2.84%	3.57%	5.75%	3.87%	3.21%	2.86%	3.92%
	$DV_k^{(2)}$	1.51%	0.54%	0.46%	0.43%	0.73%	3.91%	0.27%	0.84%	1.01%	1.51%
	DV_k	5.96%	4.11%	3.88%	3.27%	4.30%	9.66%	4.14%	4.05%	3.87%	5.43%
period C	$DV_k^{(1)}$	3.47%	3.47%	2.99%	2.64%	3.14%	6.45%	6.03%	4.79%	3.87%	5.28%
	$DV_k^{(2)}$	0.47%	0.79%	0.59%	0.26%	0.53%	1.33%	2.12%	1.50%	0.84%	1.45%
	DV_k	3.94%	4.26%	3.58%	2.90%	3.67%	7.78%	8.15%	6.29%	4.70%	6.73%
period D	$DV_k^{(1)}$	3.73%	2.50%	2.25%	2.02%	2.63%	4.48%	3.61%	2.90%	2.56%	3.39%
	$DV_k^{(2)}$	1.46%	1.25%	1.03%	1.26%	1.25%	1.17%	0.53%	0.70%	0.96%	0.84%
	DV_k	5.19%	3.76%	3.27%	3.28%	3.87%	5.65%	4.13%	3.60%	3.52%	4.23%
average	$DV_k^{(1)}$	4.26%	3.37%	3.02%	2.69%	3.33%	6.67%	5.47%	4.56%	3.75%	5.11%
	$DV_k^{(2)}$	1.04%	0.87%	0.94%	0.53%	0.85%	1.67%	1.15%	0.95%	0.92%	1.17%
	DV_k	5.30%	4.24%	3.96%	3.22%	4.18%	8.35%	6.62%	5.50%	4.67%	6.29%

The MOLD model alternates the MLPM model by only using one risk parameter. Thus, it cannot approximate the MLPM model precisely, as $DV_k^{(1)}$ shows. We cannot derive the minimized LPM by using the MOLD model when LPM value is used as the risk measure [5] .

We evaluate the relation model (Equation (19)) by $DV_k^{(2)}$. Since $DV_k^{(2)}$ is about 1% of $\sqrt[k]{LPM_k}$, or about 20% of DV_k, the relation model is a good model of showing the relationship between λ and k.

[5] Notice that the MOLD model is developed to derive an alternative linear model in a downside risk framework.

Let $\lambda_O(k)$ denote the λ value meeting $\min\limits_{\lambda}\left[\sqrt[k]{LPM(k,\lambda)}\right]$. *Table* 7 shows $\lambda(k,36)$, $\lambda_O(k)$, and the difference $\lambda_O(k)-\lambda(k,36)$. The average of six kinds of these values are listed.

Table 7: $\lambda(k,36)$, $\lambda_O(k)$, and these difference

	No Upper Limit				$U_j = 5\%$			
	$k=2$	$k=3$	$k=4$	$k=5$	$k=2$	$k=3$	$k=4$	$k=5$
$\lambda(k,36)$	0.143	0.283	0.391	0.474	0.143	0.283	0.391	0.474
$\lambda_O(k)$								
period A	0.363	0.463	0.527	0.531	0.147	0.231	0.384	0.610
period B	0.220	0.343	0.460	0.521	0.280	0.307	0.333	0.350
period C	0.135	0.364	0.543	0.561	0.063	0.338	0.348	0.407
period D	0.302	0.452	0.527	0.578	0.175	0.535	0.587	0.613
average	0.255	0.406	0.514	0.548	0.166	0.353	0.413	0.495
$\lambda_O(k)-\lambda(k,36)$								
period A	0.220	0.180	0.135	0.058	0.004	−0.052	−0.008	0.136
period B	0.077	0.060	0.069	0.047	0.137	0.024	−0.058	−0.124
period C	−0.007	0.081	0.152	0.088	−0.080	0.055	−0.043	−0.067
period D	0.159	0.169	0.135	0.105	0.032	0.252	0.196	0.140
average	0.112	0.120	0.123	0.074	0.023	0.070	0.022	0.021

※ $\lambda_O(k) = \{\lambda \mid \min\limits_{\lambda}\left[\sqrt[k]{LPM(k,\lambda)}\right]$

Figure 12 depicts $\sqrt[k]{LPM_k}$ of the MLPM model (thin lines), and 101 kinds of $\sqrt[k]{LPM(k,\lambda)}$ of the MOLD model (bold lines) for the period D, $r_E = 0.5\%$, $r_G = 0.0\%$. The top four graphs are in the case that the upper limit is not constrained, and the lower four graphs are in the case that the upper limit is 5% constrained.

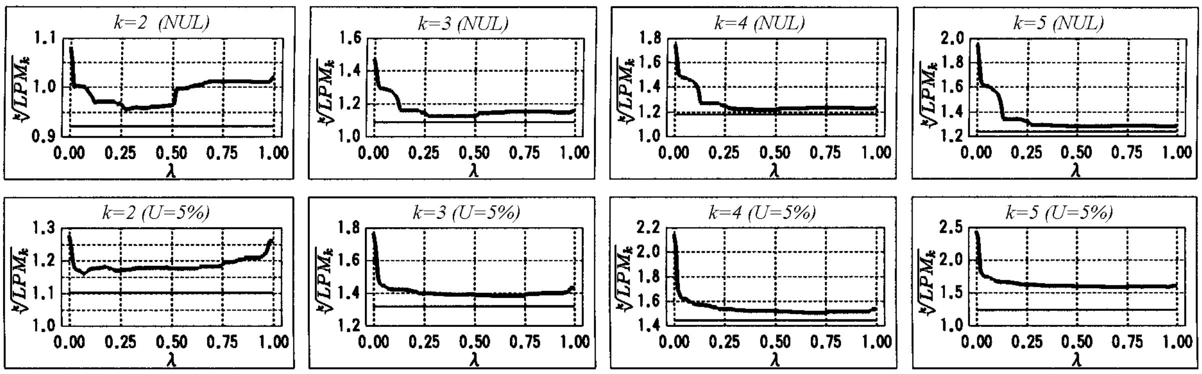

Figure 12: $\sqrt[k]{LPM_k}$(thin lines) and $\sqrt[k]{LPM(k,\lambda)}$(bold lines)

The shapes of $\sqrt[k]{LPM(k,\lambda)}$ curves are various, depending on the various periods and parameters.

6 Conclusion

In this paper, we propose the MOLD model as an alternative portfolio selection model in a downside risk framework. Since the objective function of the MOLD model is the weighted sum of the objective function of l_1 MLPM model and that of l_∞ MLPM model, the MOLD model is a linear programming model. Thus, the MOLD model is easy to solve because of the linear programming model, whereas the MOLD model is similar in risk preference to the MLPM model. The risk parameter λ of the MOLD model corresponds to k of the MLPM model. We model the relationship between λ and k.

We first analyze the characteristics of the MOLD model using historical data in Tokyo Stock Exchange. When the risk parameter λ is large, we can find the following information;

- the minimum return is large, or the maximun target shortfall is small,
- the average target shortfall is large,
- the standard deviation and the skewness tend to be large,
- the number of the periods of the maximum target shortfall tends to increase,
- the number of the periods below the target, or the probability below the target, tends to increase.

We cannot characterize the number of invested stocks and the maximum proportion.

We investigate the relationship between the risk parameter λ and the proportions invested in stocks. The proportions change drastically when $\lambda = 0$ through $\lambda = 0.1$. However, the proportions change a little when $\lambda \geq 0.1$. The information about the proportions are useful when we decide how to invest in stocks. We also show the efficient frontiers in the expected return and the OLD space.

Next, we compare the MOLD model with the MLPM model. Both return histograms look different, however the return cumulative frequency distributions of both models tend to be similar. This reason is that the number of periods to be maximum target shortfall of the MOLD model is larger than that of the MLPM model. The common proportions of both models are also similar. While both models are similar, the MOLD model has the following unique features;

(1) the maximum target shortfall of the MOLD model is smaller than that of the MLPM model,

(2) the number of periods below the target of the MOLD model is smaller than that of the MLPM model,

(3) the average target shortfall of the MOLD model tends to be smaller than that of the MLPM model,

(4) the number of periods to be maximum target shortfall of the MOLD model is larger than that of the MLPM model.

When the investment manager recognizes these features as being important in order to manage the downside risk, the MOLD model contributes to the risk management.

We compare both models using the LPM values. The MOLD model cannot approximate to the MLPM model precisely. If we approximate the MLPM model precisely to the linear programming model, we need to use the piecewise linear objective function to approximate the MLPM model, instead of using the MOLD model. However, the MOLD model can alternate the MLPM model by only one parameter.

The risk parameter λ can simply describe the detail risk preference. Equation (19) can be rearranged, and $k(\lambda, T)$ can be derived ;

$$k(\lambda, T) = \frac{\log T}{\log T - \log\{1 + \lambda \cdot (T-1)\}}. \tag{27}$$

For example, $k(0.2, 36) = 2.38$.

We test the MOLD model numerically using historical stock data, because (1) the usual portfolio selection problem is applied to stocks, and (2) the computational ability can be tested for a large scale problem. However, we need to use other prediction data instead of historical data. This is our future research in order to apply the downside risk model to selecting the stock portfolio.

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