Abstract

This paper discusses optimal dynamic investment policies for investors, who make the investment decisions in each of the asset categories over time. We construct the framework integrating stochastic optimization and Monte Carlo simulation for dynamic asset allocation, and we propose the linear programming models using simulated paths to solve a large-scale problem in practice. Linear programming models can be formulated to adopt either a fixed-value rule or a fixed-amount rule instead of the general fixed-proportion rule. These formulations can be simply implemented and solved very fast. Some numerical examples are tested to illustrate the characteristics of the models. These models can be used to improve the trade-off between risk and expected wealth, and we can get interesting results for the dynamic asset allocation policies.

1 Introduction

The investors need to maximize the expected utility of the returns from an investment portfolio, or minimize the risk of returns subject to the required expected return. They must decide the optimal investment proportion of the portfolio in securities in order to meet the investors satisfaction. This paper discusses optimal dynamic investment policies for investors, who make the investment decisions in each of the asset categories over time. This problem is called dynamic asset allocation.

The asset allocation decisions are critical for investors with diversified portfolios. Institutional investors must manage their strategic asset mix over time to achieve favorable returns subject to various uncertainties, policy and legal constraints, and other requirements. Two kinds of multi-period stochastic optimization models can be used to solve this problem, as follows;

(1) stochastic programming model using scenarios,
(2) stochastic control(dynamic stochastic programming) model using random samples.

Critical issues for stochastic modeling involve the handling of uncertainty and decision rules. The decisions have to be made independent upon knowing actual paths that will occur. Thus we must define decision variables and a set of constraints to prevent the optimization model from anticipating the future. In addition, we need the sufficient paths of uncertainties to get a better accuracy with respect to the future possible events. We describe the characteristics of two models in order.

The notion of scenarios is typically employed for modeling random parameters in the multi-period stochastic programming(MPSP) model, and the path of uncertainty is revealed as a scenario tree. This model is based on the expansion of the decision space, taking into account the conditional nature of the scenario tree. Conditional decisions are made at each node. To assume that a representative set of scenarios is constructed which covers the set of possibilities to a sufficient degree, the number of decision variables and constraints appearing in the scenario tree may grow exponentially. (See Mulvey and Ziemba [6, 7] in detail.)

On the other hand, the stochastic control model is originally proposed as the multi-period portfolio optimization model. The basic framework of this model is proposed by Merton[5] and Samuelson[8]. The stochastic control model can be used to reduce the decision space to a set of policies. In this paper, paths of uncertainty are revealed as simulated paths with any stochastic process for implementing the dynamic stochastic control numerically. We can get a better accuracy of uncertainty by using simulated paths rather than a scenario tree. However, the investment decisions at each time must be limited to a fixed-proportion rule in general. In addition, though the stochastic control model can be formulated as a stochastic programming model, this is formulated as a non-convex (non-linear) programming model, and it is difficult to solve the problem and to find the global optimum solution. Two stochastic optimization models involve trading off flexible decision rules and the accuracy of uncertainty.

In this paper, we construct the framework integrating stochastic optimization and Monte Carlo simulation for dynamic asset allocation, and we propose the alternative models using simulated paths, which can be formulated as linear programming models to solve a large-scale problem in practice. Linear programming models can be formulated to adopt either a fixed-value rule or a fixed-amount rule instead of the general fixed-proportion rule. These formulations can be simply implemented and solved very fast by using a sophisticated mathematical programming software.
The paper is organized as follows. Section 2 introduces three types of models and their formulations using simulated paths. Section 3 presents numerical examples to the MPSP model with the fixed-amount rule. Section 4 provides some concluding remarks and our future research.

2 The MPSP model using simulated paths

2.1 Preparation

An asset return is a variable parameter, and its process is expressed by a stochastic differential equation, or a time series model. We can sample the path of each asset return on each simulation trial. We call it the "simulated path". The example of the simulated path is shown in Figure 1.

![Figure 1: Simulated path](image)

Next, we show the following three types of models:
1. MPSP model with the fixed-proportion rule,
2. MPSP model with the fixed-value rule,
3. MPSP model with the fixed-amount rule.

These models have the different decision variables, or the different investment decision rule. Investment proportions are decision variables in the MPSP model with the fixed-proportion rule. The solution to the asset allocation decision of this model provides the recommended proportions. The investment decisions are various over time, however they are fixed on all paths at each time. Therefore, the investment proportions of an risky asset and cash to any path are same. The "fixed-" does not mean "buy and hold strategy", and "constant rebalance strategy".

Investment values are decision variables in the MPSP model with the fixed-value rule. The investment values of an risky asset to any path are same, however the cash is different in each path.

Investment amounts are decision variables in the MPSP model with the fixed-amount rule. The investment amounts of an risky asset to any path are same, however the cash is different in each path.

In general, we decide the investment proportions for the asset allocation decision. Therefore, we start with the MPSP model with the fixed-proportion rule using simulated path.

We invest \( n \) risky assets and cash. The investment is made at time 0(present), and time \( T \) is the planning horizon.

2.2 Modeling with the fixed-proportion rule

Notations used in this model are as follows.

(1) Parameters

\( I \) : number of the simulated path

\( \mu_{jt}^{(i)} \) : rate of return of asset \( j \) of path \( i \) in period \( t \),

\( j = 1, \ldots, n; t = 1, \ldots, T; i = 1, \ldots, I \)

\( r_{t-1}^{(i)} \) : interest rate of path \( i \) in period \( t \) (the rate at time \( t-1 \) is used), \( t = 1, \ldots, T; i = 1, \ldots, I \)

\( W_0 \) : initial wealth

\( W_G \) : target wealth at the planning horizon

\( W_E \) : required expected wealth at the planning horizon

(2) Decision variables

\( w_{jt} \) : investment proportion of asset \( j \) at time \( t \),

\( j = 1, \ldots, n; t = 0, \ldots, T-1 \)

\( c_t \) : cash ratio at time \( t \), \( t = 0, \ldots, T-1 \)

\( W_t^{(i)} \) : wealth of path \( i \) at time \( t \),

\( t = 1, \ldots, T; i = 1, \ldots, I \)

\( q^{(i)} \) : shortfall below the target wealth of path \( i \) at the planning horizon, \( i = 1, \ldots, I \)

We formulate the MPSP model with the fixed-proportion rule as follows.

(1) Investment decision at time 0 : \( w_{j0} \), \( c_0 \)

\[
\sum_{j=1}^{n} w_{j0} + c_0 = 1
\]

(2) Wealth of path \( i \) at time 1, \( i = 1, \ldots, I \)

\[
W_1^{(i)} = \left\{ \sum_{j=1}^{n} \left( 1 + \mu_{jt}^{(i)} \right) w_{jt0} + (1 + r_{t0})c_0 \right\} W_0
\]

(3) Wealth of path \( i \) at time \( t \), \( t = 2, \ldots, T; i = 1, \ldots, I \)

\[
W_t^{(i)} = \left\{ \sum_{j=1}^{n} \left( 1 + \mu_{jt}^{(i)} \right) w_{jt-1} + (1 + r_{t-1}^{(i)}) c_{t-1} \right\} W_{t-1}^{(i)}
\]

\[
(\text{decision at time } t-1) \sum_{j=1}^{n} w_{jt-1} + c_{t-1} = 1
\]
(4) Terminal wealth, \((i = 1, \ldots, I)\)
\[ W_T^{(i)} = \left\{ \sum_{j=1}^{n} \left(1 + \mu_{jT}^{(i)}\right) w_{j,T-1} + \left(1 + r_{T-1}^{(i)}\right) v_{T-1}^{(i)} \right\} W_{T-1}^{(i)} \]

(5) Objective function
The shortfall below the target wealth (the lower partial moment \([1, 3]\)) at the planning horizon is minimized, subject to the requirement of the minimum expected terminal wealth.

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{T} \sum_{i=1}^{I} q^{(i)} \\
\text{subject to} & \quad W_T^{(i)} + q^{(i)} \geq W_G, \quad (i = 1, \ldots, I) \\
& \quad q^{(i)} \geq 0, \quad (i = 1, \ldots, I) \\
& \quad \frac{1}{T} \sum_{i=1}^{I} W_T^{(i)} \geq W_E 
\end{align*}
\]

This model is too difficult to find the global optimum solutions because Equations for (3) and (4) are non-linear and non-convex constraints.

2.3 Modeling with the fixed-value rule

The investment values are decided instead of the investment proportions in this model. The global optimum solutions can be easily derived in practice, because the model is formulated as a linear programming model.

Added notations in this model are as follows;
\[ x_{jt} : \text{investment value of asset } j \text{ at time } t, \]
\[ (j = 1, \ldots, n; \ t = 0, \ldots, T - 1), \]
\[ v_0 : \text{cash at time } 0, \]
\[ v_t^{(i)} : \text{cash of path } i \text{ at time } t, \]
\[ (t = 0, \ldots, T - 1; \ i = 1, \ldots, I). \]

We formulate the MPSP model with the fixed-value rule as follows \(^2\):

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{T} \sum_{i=1}^{I} q^{(i)}, \quad (1) \\
\text{subject to} & \quad \sum_{j=1}^{n} x_{jt} + v_0 = W_0, \quad (2) \\
& \quad \sum_{j=1}^{n} \left(1 + \rho_{jT}^{(i)}\right) x_{j,T-1} + \left(1 + r_{T-1}^{(i)}\right) v_{T-1}^{(i)} = \sum_{j=1}^{n} x_{jt} + v_t^{(i)}, \quad (i = 1, \ldots, I), \quad (4) \\
& \quad \sum_{j=1}^{n} \left(1 + \rho_{jT}^{(i)}\right) x_{j,t-1} + \left(1 + r_{t-1}^{(i)}\right) v_{t-1}^{(i)} = \sum_{j=1}^{n} x_{jt} + v_t^{(i)}, \quad (t = 2, \ldots, T - 1; \ i = 1, \ldots, I), \quad (5) \\
& \quad \sum_{j=1}^{n} \left(1 + \rho_{jT}^{(i)}\right) x_{j,T-1} + \frac{1}{T} \sum_{i=1}^{I} \left(1 + r_{T-1}^{(i)}\right) v_{T-1}^{(i)} \geq W_E, \quad (6) \\
& \quad \sum_{j=1}^{n} \left(1 + \mu_{jT}^{(i)}\right) x_{j,T-1} + \left(1 + r_{T-1}^{(i)}\right) v_{T-1}^{(i)} \geq W_G, \quad (i = 1, \ldots, I), \quad (7) \\
& \quad x_{jt} \geq 0, \quad (j = 1, \ldots, n; \ t = 0, \ldots, T - 1), \quad (8) \\
& \quad v_0 \geq 0, \quad (9) \\
& \quad v_t^{(i)} \geq 0, \quad (t = 1, \ldots, T - 1; \ i = 1, \ldots, I), \quad (10) \\
& \quad q^{(i)} \geq 0, \quad (i = 1, \ldots, I), \quad (11)
\end{align*}
\]

where \(\mu_{jT}^{(i)} = \frac{1}{T} \sum_{i=1}^{I} \rho_{jT}^{(i)}\).

2.4 Modeling with the fixed-amount rule

The investment value multiplies the price by the amount.

\[
\begin{align*}
\text{Investment value} & = \text{price (per unit)} \times \text{amount (units)} \\
& = \frac{\text{price}}{\text{base value}} \times \text{base value (per unit)} \times \text{amount (units)} \\
& = \text{relative price} \times \text{investment base value}
\end{align*}
\]

The investment value can be divided into two parts, the relative price and investment base value. The relative price can be defined as the ratio of the price to the base value. We can set the base value using the face value, the current value, and so on.

New notations are introduced as follows;
\[ \rho_{j0} : \text{relative price of asset } j \text{ at time } 0, \quad (j = 1, \ldots, n), \]
\[ \rho_{jt}^{(i)} : \text{relative price of asset } j \text{ of path } i \text{ at time } t, \]
\[ (j = 1, \ldots, n; \ t = 1, \ldots, T; \ i = 1, \ldots, I), \]
\[ z_{jt} : \text{investment amount of asset } j \text{ at time } t, \]
\[ (j = 1, \ldots, n; \ t = 0, \ldots, T - 1). \]

We formulate the MPSP model with the fixed-amount rule as follows;

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{I} \sum_{i=1}^{I} q^{(i)}, \quad (12) \\
\text{subject to} & \quad \sum_{j=1}^{n} \rho_{j0} z_{j0} + v_0 = W_0, \quad (13) \\
& \quad \sum_{j=1}^{n} \rho_{jT}^{(i)} (z_{jt} - z_{j0}) = (1 + r_0) v_0 - v_t^{(i)}, \quad (i = 1, \ldots, I), \quad (14) \\
& \quad (i = 1, \ldots, I), \quad (15)
\end{align*}
\]
\[ \sum_{j=1}^{n} p_{jt}^{(i)} (z_{jt} - z_{j,t-1}) = \left( 1 + r_{t-1}^{(i)} \right) v_{j,t-1}^{(i)} - v_{i,t}^{(i)}, \]
\[ (t = 2, \ldots, T - 1; \ i = 1, \ldots, I), \quad (i = 1, \ldots, I). \]
\[ \sum_{j=1}^{n} p_{jT} z_{j,T-1} + \frac{1}{2} \sum_{i=1}^{I} \left( 1 + r_{T-1}^{(i)} \right) v_{j,T-1}^{(i)} \geq W_F, \quad (i = 1, \ldots, I). \]
\[ \sum_{j=1}^{n} p_{jT} z_{j,T-1} + \left( 1 + r_{T-1}^{(i)} \right) v_{j,T-1}^{(i)} + q^{(i)} \geq W_G, \quad (i = 1, \ldots, I). \]

2.5 Model selection

Three MPSP models using simulated paths are not equivalent, while three MPSP models using scenarios are equivalent. Therefore, we can select the model in correspondence with the trading strategy. If we do not have to take the trading strategy with the fixed-proportion rule, we should adopt the trading strategy with the fixed-value rule or with the fixed-amount rule because we can easily solve these models.

2.6 Problem size

A problem size is as in Table 1.

<table>
<thead>
<tr>
<th>constraints</th>
<th>fixed-proportion</th>
<th>fixed-value</th>
<th>fixed-amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T + 1)(I + 1)</td>
<td>TI + 2</td>
<td>TI + 2</td>
<td></td>
</tr>
<tr>
<td>decision variables</td>
<td>(T + 1)I + (n + 1)T</td>
<td>(n + I)T + 1</td>
<td>(n + I)T + 3</td>
</tr>
</tbody>
</table>

† Non-negative constraints are excluded.

The problem size is very large, but it is very sparse. The relationship between the number of path and non-zero elements in the constraints to the MPSP model with the fixed-amount rule is as in Figure 2.

3 Numerical Examples

We test the model using numerical examples. Some cases for the MPSP model with the fixed-amount rule are tested. Three periods model is solved, and one period is one month in this example. The number of simulated paths is 500. The number of constraints is 1,535 except non-negative constraints, and the number of decision variables is 1,518. All of the problems are solved using NUOPT. Computing time of this

The rate of return \( \mu_{jt}^{(i)} \) is generated as follows.

1. The rate of return of asset \( j \) derived at time \( t \) is normally distributed with mean \( \mu_{jt} \) and standard deviation \( \sigma_{jt} \), and it is generated as follows;
\[ \mu_{jt}^{(i)} = \mu_{jt} + \sigma_{jt} \epsilon_{jt}^{(i)}, \]
where \( \epsilon_{jt}^{(i)} \) is a random sample from a multi-variate standardized normal distribution.

2. The random variable \( \epsilon_{jt} \) for \( j = 0, \ldots, n; \ t = 1, \ldots, T \) follows that
\[ \epsilon_{jt} \sim N(\mu_{jt}, \Sigma), \]
where \( \Sigma \) is \( (n + 1)T \times (n + 1)T \) correlation matrix.

Asset 0 is the interest rate, and \( \mu_{0t}^{(i)} \) is the change rate of interest rate. The call rate \( r_{0t}^{(i)} \) is calculated as follows;
\[ r_{0t}^{(i)} = r_0 \times \left( 1 + \mu_{0t}^{(i)} \right), \]
\[ r_{jt}^{(i)} = r_{j,t-1}^{(i)} \times \left( 1 + \mu_{0t}^{(i)} \right), \quad (t = 2, \ldots, T - 1). \]

Initial (relative) prices of stock, bond, and CB assume to be 1 for simplicity. The initial call rate is
0.44%. The initial wealth and the target wealth are 100 million yen.

We test eight cases where the objective functions and their related constraints to the required expected terminal wealth are various as in Table 2.

Table 2: Test case

<table>
<thead>
<tr>
<th>Case</th>
<th>Objective</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>minimization of the risk</td>
</tr>
<tr>
<td>2</td>
<td>minimization of the risk subject to $W_T = 10,165$</td>
</tr>
<tr>
<td>3</td>
<td>minimization of the risk subject to $W_T = 10,180$</td>
</tr>
<tr>
<td>4</td>
<td>minimization of the risk subject to $W_T = 10,195$</td>
</tr>
<tr>
<td>5</td>
<td>minimization of the risk subject to $W_T = 10,210$</td>
</tr>
<tr>
<td>6</td>
<td>minimization of the risk subject to $W_T = 10,225$</td>
</tr>
<tr>
<td>7</td>
<td>minimization of the risk subject to $W_T = 10,240$</td>
</tr>
<tr>
<td>8</td>
<td>maximization of the expected terminal wealth</td>
</tr>
</tbody>
</table>

Table 3 shows the investment amounts, which are the solutions of this model. In case 1, because the risk is minimized, investors have more cash and bond rather than stock and CB. While investors have only stock in case 8, because the return is maximized. In case 7, investors have CB and stock because CB and stock are more risky assets than cash or bond. These results are intuitively appealing. We can get interesting results for the dynamic asset allocation policies.

Table 4 and Table 5 show the average investment values, and the average investment proportions, which are calculated using the recommended amounts in Table 3. The expected wealth and $LPM_1$ value are also computed as in Table 6. The expected terminal wealth is the return measure, and $LPM$(lower partial moments) is the risk measure in this model. We can find the trade-off between risk and return in Table 6.

Figure 3 is the cumulative distribution of the terminal wealth. The terminal wealth on each path is sorted by value, and it can be depicted as in Figure 3. We find that the larger the required expected terminal wealth is, the larger the volatility is. We can control the risk and return of the terminal wealth in some degree directly using this model.

4 Conclusion

In this paper, we propose the multi-period linear stochastic programming model for the investment decision of the dynamic asset allocation policies. When the decision is the investment proportion, the stochastic programming model proposed in this study has the equivalent formulation to the stochastic control model, or dynamic stochastic programming model. However, this model is difficult to solve and to find the global optimum solutions. The models with the fixed-value rule and the fixed-amount rule can be formulated as the linear programming models. Thus, it can be solved very fast for the large scale problem in practice. The model with the fixed-amount rule is examined with numerical test. The trade-off relationship between return and risk, and the investment decision to each case is examined. The results are intuitively appealing. In our future research, we must investigate the characteristics of these models using numerical tests.

References

Table. 3 : Investment amounts

<table>
<thead>
<tr>
<th>case (average)</th>
<th>stock</th>
<th>bond</th>
<th>CB</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7849.5</td>
<td>7865.2</td>
<td>6984.1</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
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<td>4</td>
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<td>5</td>
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<td>6</td>
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<td>8</td>
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</tbody>
</table>

Table. 4 : Investment values (average)

<table>
<thead>
<tr>
<th>case (average)</th>
<th>stock(average)</th>
<th>bond(average)</th>
<th>CB(average)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
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<tr>
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<td>8</td>
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</table>

Table. 5 : Investment proportions (average)

<table>
<thead>
<tr>
<th>case (average)</th>
<th>stock(average)</th>
<th>bond(average)</th>
<th>CB(average)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td></td>
<td></td>
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<tr>
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Table. 6 : Expected wealth and $LPM_1$ value

<table>
<thead>
<tr>
<th>LPM$_1$</th>
<th>expected wealth</th>
<th>case (average)</th>
</tr>
</thead>
<tbody>
<tr>
<td>case 1</td>
<td>10000.0</td>
<td>0.00%</td>
</tr>
<tr>
<td>case 2</td>
<td>10000.0</td>
<td>0.00%</td>
</tr>
<tr>
<td>case 3</td>
<td>10000.0</td>
<td>0.00%</td>
</tr>
<tr>
<td>case 4</td>
<td>10000.0</td>
<td>0.00%</td>
</tr>
<tr>
<td>case 5</td>
<td>10000.0</td>
<td>0.00%</td>
</tr>
<tr>
<td>case 6</td>
<td>10000.0</td>
<td>0.00%</td>
</tr>
<tr>
<td>case 7</td>
<td>10000.0</td>
<td>0.00%</td>
</tr>
<tr>
<td>case 8</td>
<td>10000.0</td>
<td>0.00%</td>
</tr>
</tbody>
</table>

Figure. 3 : Cumulative distribution of the terminal wealth