Dynamic Asset and Liability Management Models for Pension Systems
– The Comparison between Multi-period Stochastic Programming Model and Stochastic Control Model –

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1 Introduction

How to invest pension assets is the most important issue in maintenance and management of corporate pension systems. The managers of pension system have to decide how to allocate and how to manage their assets in order to guarantee future payments to the pensioners. This paper discusses optimal dynamic investment decisions in consideration of pension liabilities over time for pension managers. This problem is called "dynamic asset and liability management".

We can mainly use two types of optimization models for long-term portfolio choice with rebalancing: stochastic control model in mathematical finance and multi-period stochastic programming model in financial engineering. There are a lot of studies in the literature, see Campbell and Viceira(2002) for stochastic control models and Ziemba and Mulvey(1998), Zenios and Ziemba(2007) for multi-period stochastic programming models. However, to our knowledge, these are studied in each area [1] , and even computing results from these research are not compared. There is a strong focus on finding the implications for the problem in the stochastic control model, because it is difficult to derive the analytical solutions in general and even numerical solutions for the complicated problems, and we need to solve the problem in the simple setup. On the other hand, there is a strong focus on solving the practical problem including several constraints and a goal in a real world to make or support the management decision. Therefore, we need to formulate the model with the approximate description of states of investment decision and uncertainties of asset prices. As just described, it is not easy to compare these models because they focus on the different points and the models are described in the different setup.

Our final goal is to combine the advantages of two models, and to develop the practical model and the solution method for pension systems. In this paper, we start to examine the difference of two models in the simple setup to achieve our goal.

The multi-period stochastic programming model is called a discrete model because of discrete-time, investment decision in discrete states and discrete scenarios or path of asset prices. On the other hand, the stochastic control model is called a continuous model because of continuous-time, investment decision in continuous states and continuous distributions of asset prices.

Table 1 shows the difference between the discrete model and the continuous model [2] .

We explain the pros and cons of two models. Multi-period stochastic programming model is desirable from the viewpoints of time interval and flexibility of the model representation, because a policy asset mix is updated annually for pension systems and the model must be solved with the

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[1] Hochreiter, Pflug and Paulsen(2007) model the unit-linked life insurance contracts with guarantees, and solve the optimal problem. Discrete-time model and continuous-time model are solved respectively in the different setup, but the solutions are not compared.

[2] Table 1 contains only linear programming problem as discrete model, and only the problem that we can obtain an analytic solution as continuous model.

Table 1: Difference of a discrete model and a continuous model

	multi-period stochastic programming model	stochastic control model
time	discrete	continuous
investment decision	discrete state at discrete time	continuous state at continuous time
distribution of asset price	discrete scenarios or paths	continuous distribution
model description	flexible	limited
objective function	LPM, CVaR	utility function, LPM(1), CVaR

practical constraints in a real world. However, we need to increase states of investment decision to make conditional decisions and increase the number of scenarios or paths to describe the distribution of asset prices accurately. This incurs the computational limitation because we need to solve the large-scale mathematical programming problem. On the other hand, stochastic control model is desirable from the viewpoint of the representation of continuous states [3] where there is a continuum of possible choice(investment decision). It is also the powerful approach because it may be able to give the analytical solution. However, whether the analytical solution might be obtained or not depends on the setup of the problem and continuous trading (rebalancing) of a policy asset mix is unreal for pension systems.

The risk on pension system management is that the pension asset falls below pension liability, because it is the most important for pensioners to guarantee their pension benefits, and it might be natural to decide the investment policy so that their asset should not fall below their liability.

One of the well-known downside risk measure below target is lower partial moment (LPM), and it can be applied to the pension ALM problem. We can select the alternative risk measures to solve the stochastic programming model [4] . However, there are few risk measures which can be selected because it is difficult to derive the analytical solution(closed form solution) by using the stochastic control model. Fortunately, Civitanic and Karatzas(1999) shows the analytical solution by the martingale approach in the stochastic control model. We select the stochastic programming model using simulated paths proposed by Hibiki(2001,2003) because we can describe uncertainties more accurately than would the scenario tree. It is called a hybrid model. Hibiki(2006) shows that the hybrid model can evaluate and control risk better than the scenario tree model. We can easily generate simulated paths, and use to compare it with the stochastic control model. We have three kinds of discretizations in the discrete model; discretization of time, state and distribution. However, we focus on time and state discretizations associated with investment decision, and examine the difference between a continuous model and a discrete model in this paper. The reason is both state and distribution are continuous in the continuous model and it is difficult to compare two models only for alternative discretization. It is expected that the different solutions are derived from two models. We analyze the sensitivity of some parameters, such as the number of time steps, in order to clarify the reason that the difference occurs.

This paper is organized as follows. In section 2, we describe a continuous model and a discrete model of pension ALM problem in detail. Section 3 presents a part of the numerical results. Section 4 shows the conclusion and outlines our future research.

2 The multi-period optimization problem considering pension liability

We formulate the model to minimize LPM with the target for the pension asset which is equal to pension liability value at time T : C. The minimization of the expected target deviation of the

[3] Asset prices can be represented with a continuous distribution.

[4] We often solve the problem that maximizes the expected value of the utility function of surplus wealth(the amount of pension asset minus the amount of pension liability). However, we need much computing time to solve the large-scale stochastic programming problem with a non-linear utility function.

terminal pension asset value $W(T)$ can be represented as:

$$\min\ \mathbf{E}[(C - W(T))^+], \tag{1}$$

where a function $(x)^+ = x$ $(x \geq 0)$, 0 $(x < 0)$. The amount of pension liability C is constant. We assume that we invest in a risky asset and a riskless asset to clarify the difference of models. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carring an a dimensional Brownian motion $(w_t)_{t\in[0,T]}$ is given. The stochastic process of a risky asset $S(t)$ and a riskless asset $B(t)$ are

$$\begin{aligned} dS(t) &= S(t)(\mu dt + \sigma dw(t)), & (2)\\ dB(t) &= B(t) r dt, & (3) \end{aligned}$$

where μ and σ denote the mean and standard deviation of $S(t)$, r is a risk-free interest rate, and $w(t)$ is one-dimensional Brownian motion.

If $W(0) \geq e^{-rT}C$, the optimal solution is to invest a full amount of their pension asset in a riskless asset because their pension asset value never falls below their pension liability value as long as they invest only in a riskless asset. It's a trivial, and therefore we set $W(0)e^{rT} \leq C$.

2.1 Continuous model

Civitanic and Karatzas(1999) shows the solution of the above optimization problem by the martingale approach. The pension asset value under the optimal strategy at time T is [5]

$$W^*(T) = W_G \mathbf{1}_{\{\xi(T)\square\bar{\xi}\}}, \tag{4}$$

where $\xi(t)$ is state price density process and $\bar{\xi}$ satisfies

$$W(0) = W_G \mathbf{E}[\xi(T)\mathbf{1}_{\{\xi(T)\square\bar{\xi}\}}]. \tag{5}$$

The process of the pension asset value under the optimal strategy is as follows: [6]

$$\begin{aligned} W^*(t) &= W_G e^{-r(T-t)}\Phi\left(\frac{D(t)}{\sqrt{T-t}}\right), & (6)\\ D(t) &= \sqrt{T}\Phi^{-1}\left(\frac{W(0)}{W_G e^{-rT}}\right) + \frac{1}{\sigma}\log\left(\frac{S(t)}{S(0)}\right) - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)t, & (7) \end{aligned}$$

where $\Phi(x)$ is the standard normal distribution function, namely $\Phi(x) = \int_{-\infty}^{x} e^{-\frac{x^2}{2}} dx$.

The saving ratio $Y(t)$ of pension system is defined as $Y(t) = \frac{W(t)}{W_G e^{-r(T-t)}}$, the investment ratio to a risky asset $\pi(t)$ is represented as

$$\pi(t, Y(t)) = \begin{cases} \frac{1}{\sigma\sqrt{T-t}Y(t)}(\varphi \circ \Phi^{-1})(Y(t)) & 0 < Y(t) < 1 \\ 0 & Y(t) \geq 1 \end{cases} \tag{8}$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

If we can invest in d-risky asset and a riskless asset, we can also solve the optimization problem (2). The investment ratios among d risky assets of the optimal portfolio becomes equal with the investment ratios of myopic portfolio [7]. Then, when we consider d-risky assets myopic portfolio to be a new asset, the optimization problem on a risky asset is consistent with the optimization problem on d risky assets.

2.2 Discrete model

We formulate the hybrid model in Hibiki(2006) as the discrete model. Discrete values of asset returns are generated by Monte Carlo simulation. Then the generated paths on which the same investment decision is made are bundled because model solves anticipating the event in the future. After that, we calculate optimal investment strategy on the set of bundled paths. The optimal

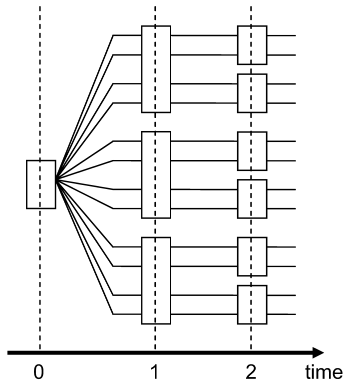

Figure 1: Sets of simulation paths

investment strategy given by this model consists of how to bundle paths and investment amount of assets.

Figure 1 shows sets of simulation paths in hybrid model. Hybrid model does not allow that paths that have passed other decision states once pass the same decision state. Because investment decisions must be similar in similar state, we want to bundle paths by pension asset value. In this paper, we bundle paths equally. (If paths are divided to n sets, each set has $1/n$ paths.) But the pension asset value depends on investment strategy that is going to be obtained. Then, we repeat bundling paths and optimizing investment strategy until the obtained strategy converges.

Next we explain how to optimize investment strategy. Notations are as follows. The period to time horizon T is divided into N steps, we assume $t = 0, t_1, \ldots, t_N (t_N = T)$, and $t_k = k\frac{T}{N}$.

1. Sets

   V_t : set of fixed-decision nodes at time t, $(s \in V_t)$
   V_t^s : set of paths passing any fixed-decision node s at time t, $(t = t_1, \ldots, t_N; i \in V_t^s)$

2. Parameters

   I : number of simulated paths
   S_0 : price of a risky asset at time 0
   $S_t^{(i)}$: price of a risky asset of path i at time t $(t = t_1, \ldots, t_N; i = 1, \ldots, I)$
   r : rate of return of a riskless asset(risk-free interest rate)
   W_0 : initial pension asset value at time 0
   W_G : target for pension asset value (pension liability value) at time T

3. Decision variables

   z_0 : investment unit for a risky asset and time 0
   z_t^s : investment unit for a risky asset, time t and state s $(t = t_1, \ldots, t_N; s \in S_t)$
   $W_t^{(i)}$: pension asset value for path i and time t $(t = t_1, \ldots, t_N; i = 1, \ldots, I)$
   $q^{(i)}$: shortfall below target for terminal pension asset value of path i $(i = 1, \ldots, I)$
   v_0 : cash at time 0
   $v_t^{(i)}$: cash of path i and time t $(t = t_1, \ldots, t_N; i = 1, \ldots, I)$

[5] $W^*(T)$ is not necessarily unique.

[6] Civitanic and Karatzas(1999) presents the solution of minimal LPM strategy are the same as that of minimal shortfall probability strategy, if W_G is constant.

[7] The myopic portfolio is the optimal portfolio for the investors who has log utility function.

The hybrid model is formulated as follows [8] :

$$\textbf{Minimize} \quad \frac{1}{I}\sum_{i=1}^{I} q^{(i)}, \tag{9}$$

subject to

$$\begin{aligned}
& S_0 z_0 + v_0 = W_0, \\
& S_{t_1}^{(i)} h^{(i)}(z_0) + (1 + r(t_1 - t_0))v_0 = S_{t_1}^{(i)} h^{(i)}(z_{t_1}^s) + v_{t_1}^{(i)} \quad (i \in V_{t_1}^s; s \in V_{t_1}), \\
& S_{t_k}^{(i)} h^{(i)}(z_{t_{k-1}}^{s'}) + (1 + r(t_k - t_{k-1}))v_{t_{k-1}}^{(i)} = S_{t_k}^{(i)} h^{(i)}(z_{t_k}^s) + v_{t_k}^{(i)} \\
& \qquad\qquad\qquad\qquad (k = 2, \ldots, T-1; i \in V_{t_k}^s; s \in V_{t_k}; s' \in V_{t_{k-1}}), \\
& W_{t_N}^{(i)} = S_{t_N}^{(i)} h^{(i)}(z_{t_{N-1}}^{s'}) + (1 + r(t_N - t_{N-1}))v_{t_{N-1}}^{(i)} \quad (i \in V_{t_{N-1}}^s; s' \in V_{t_{N-1}}),
\end{aligned} \tag{10}$$

$$W_{t_N}^{(i)} + q^{(i)} \geq W_G \quad (i = 1, \ldots, I). \tag{11}$$

The function $h^{(i)}(z_t^s)$ in the set of constraints (10) is called the investment unit function. Using the investment unit function, the formulation (9) and (10) can represent the problem with fixed-unit strategy, fixed-amount strategy and fixed-proportion strategy. When we solve the problem with fixed-unit strategy and fixed-proportion strategy, we define $h^{(i)}(z_t^s) = z_t^s$ and $h^{(i)}(z_t^s) = \left(\frac{W_t^{(i)}}{S_t^{(i)}}\right) z_t^s$ respectively. But the optimization problem with fixed-proportion strategy is non-convex and non-linear program because $W_t^{(i)}$ in the investment unit function are complementary variables and described as decision variables in the stochastic programming model. Then, we solve the problem by using the iterative algorithm. At first, we solve the problem with fixed-unit strategy. Next, we assign the optimal pension asset value $W_t^{(i)}$ to the investment unit function, and we solve the problem with fixed-proportion strategy approximately. After that, the approximate problems with fixed-proportion strategy are solved iteratively, updating the value $W_t^{(i)}$, until the objective function values (9) converge or become lower than a tolerance.

3 Numerical results

We calculate the optimal strategy to examine the differences of two types of the models by the following three methods. We compare the terminal pension asset value on the simulated paths calculated by the following three methods [9].

1. Continuous-time rebalancing method with the strategy of the stochastic control model (4) [10].

2. Discrete-time rebalancing method with the strategy of the stochastic control model or Equation (8) [11].

[8] The optimal solution derived by minimizing the first order LPM is not always unique. We include the term of maximizing average terminal pension asset value in the objective function because we want to derive the unique optimal solution of the hybrid model. The following objective function is used in the numerical tests:

$$\textbf{Minimize} \quad \frac{1}{I}\sum_{i=1}^{I} q^{(i)} - \gamma \sum_{i=1}^{I} W_T^{(i)}.$$

We set $\gamma = 10^{-5}$ in the numerical tests to block the effect of this term.

[9] Continuous-time rebalancing method adopts trading in the continuous time. On the other hand, discrete-time rebalancing method adopts trading at the discrete time-point, and the buy and hold within the discrete time-point.

[10] The amounts of the terminal pension asset value are derived by the optimal strategy implicitly. We can calculate them directly by using the relationship between terminal price of a risky asset and terminal pension asset value or Equation (4).

[11] The terminal pension asset value in Method 2 is calculated as follows. The period to time horizon T is divided into N steps, we assume $t = 0, t_1, \cdots, t_N (t_N = T)$, and $t_k = k\frac{T}{N}$. The investment ratio for risky asset $\pi_t^{(i)}$ is given by Equation (8). If the price of risky asset $S_t^{(i)}$ $(i = 1, \cdots, I; t = t_0, t_1, \cdots, t_N)$ in path i are generated, pension asset value $W_t^{(i)}$ in path i is calculated by the following recurrence equation (12).

$$W_{t_k}^{(i)} = [1 + (t_k - t_{k-1})r(1 - \pi_{t_{k-1}}^{(i)}) + (S_{t_k}^{(i)}/S_{t_{k-1}}^{(i)} - 1)\pi_{t_{k-1}}^{(i)}]W_{t_{k-1}}^{(i)} \quad (i = 1, \cdots, I; k = 1, \cdots, N). \tag{12}$$

3. Discrete-time rebalancing method with the strategy of the multi-period stochastic programming model in Section2.2.

The difference of the discretization can be confirmed, using two methods among three methods.

- Time discretization : Method 1 vs. Method 2
  The difference of time discretization is examined by the difference between continuous-time and discrete-time rebalancing (investment decision), using the same solution derived by the stochastic control model. If the time is discretized (the number of time steps increases) unlimitedly in Method 2, it is expected that the terminal pension asset value converge to the amounts in Method 1.

- State discretization : Method 2 vs. Method 3
  The difference of state discretization is examined by the difference between the solution at a continuum of possible decision states and the solutions at possible discrete decision states.

The basic settings are as follows. If the settings may be changed, additional explanation is added.

- rate of return of a riskless asset : $r = 1\%$
- rate of return of a risky asset : mean $\mu = 5\%$, standard deviation $\sigma = 10\%$
- time horizon : $T = 10$ (years)
- number of simulated paths : $I = 5,000$(paths)
- initial pension asset value: 452.6(saving ratio [12] : 80%)
- target for pension asset value(terminal pension liability value) : $W_G = 500$

3.1 Difference caused by time discretization

We vary the number of time steps in Method 2, to see the influence by time discretization. Figure 2 shows the LPM of terminal pension asset value and the probability that the pension asset value exceed the pension liability value of 49,980 sample paths [13]. "∞" or continuous time steps is derived from Method 2.

The left-hand side of Figure 2 indicates that LPM is decreasing as time steps increase. The middle of Figure 2 shows that average of pension asset values are decreasing until 30-50 time steps, and it keeps the same value over 50 time steps. In the right-hand side of Figure 2, since there are a lot of paths which pension asset values are just below pension liability values in Method 2, the probability that pension asset value exceeds pension liability value in Method 2 is much smaller than the probability in Method 1. The probability that pension asset below the pension liability becomes almost zero in Method 1 because the continuous trading is executed with the optimal strategy derived from the stochastic control model to make pension asset and pension liability equal. The probability in Method 2 does not converge to the probability in Method 1 because the discrete trading is executed despite that the same optimal strategy is used. Figure 3 shows the distribution of terminal pension asset value. The right-hand side of Figure 3 is magnified view below 40% of cumulative probability in the left-hand side. The distributions of terminal pension asset value in Method 2 converge to the distributions in Method 1 as the number of time steps increases. The probability of pension asset below pension liability in Method 2 does not converge to the probability in Method 1 because the shape of cumulative probability function in Method 1 is bending extremely at the target liability, but the distribution of pension asset in Method 2 converges to the distribution in Method 1.

Next, we show the relationship between risky asset return and the difference of pension asset value in Figure 4 to explain the reason the pension asset values falls just below their target. The

[12] The saving ratio is the pension asset value divided by the present value of target for pension asset. The saving ratio is $\frac{500}{(1+0.01)^{10}} \times 80\% = 452.6$ in the example.

[13] Because the pension asset value in 20 paths are extremely large, they are removed from 50,000 simulated sample paths.

horizontal axis indicates the return of risky asset in the first period, from $t = 0$ to $t = 10/3$. The vertical axis indicates the difference of pension asset value in Method 2 from in Method 1(given by Equation (7)) at the end of the first period, or $t = 10/3$. The pension asset value in Method 2 is less than in Method 1 for small absolute return. This result shows the optimal strategy of the stochastic control model accumulates small return by continuous rebalancing.

For example, if the risky asset price drops, the optimal strategy of the stochastic control model is to increase investment ratio in risky asset. The optimal strategy of Method 1 is contrarian strategy when the objective function is LPM. Then, if the risky asset price reverts to the initial price, this strategy obtains more profit earned from rise in prices than loss incurred by decline in prices. This effect disappears in Method 2 since the investment ratio in risky asset is fixed during a period.

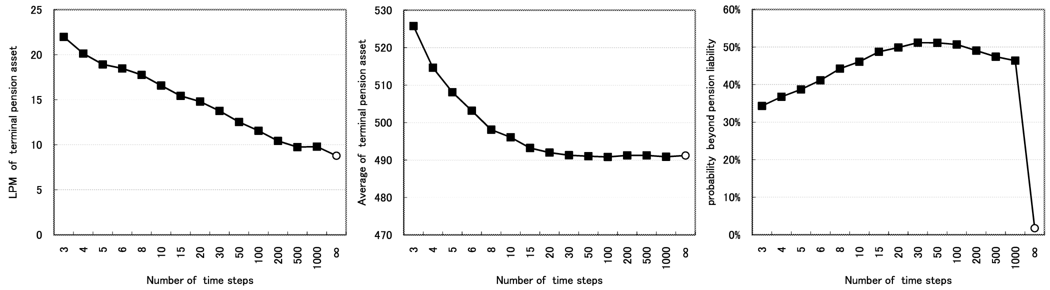

Figure 2: Difference caused by time discretization : statistics of terminal pension asset value

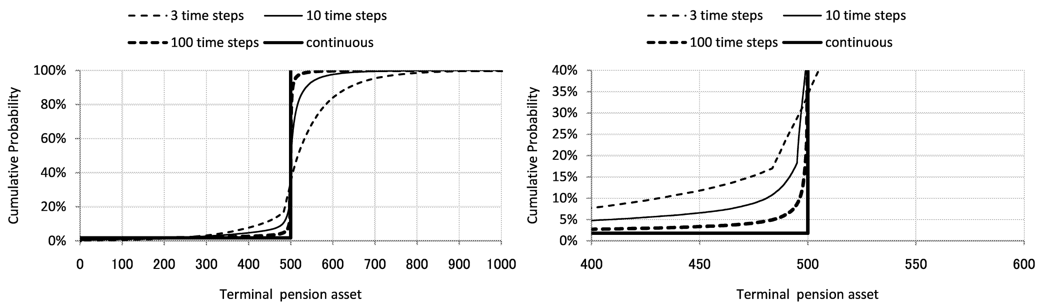

Figure 3: Difference caused by time discretization : distribution of teminal pension asset value

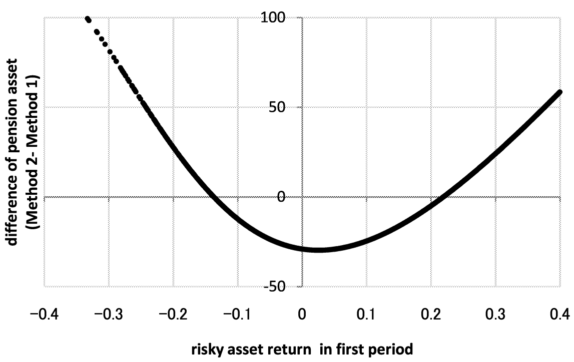

Figure 4: Relationship between pension asset value and return of risky asset at the end of the first period in 3 time steps ($t = 10/3$)

We can execute the optimal strategy that the pension asset value just exceeds the target for their pension asset with high probability in Method 1. However, above-mentioned accumulation of small return during a period disappears if we execute the discrete rebalancing strategy as in Method 2 using the optimal strategy given by continuous model. As a result, the pension asset values falls just below their target in Method 2.

3.2 Difference caused by state discretization

We compare Method 2 with Method 3 at 3 time steps in order to examine the influence of state discretization. Figure 5 shows LPM calculated by Method 2 and Method 3. The horizontal axis of this graph expresses the number of decision nodes in Method 3. (For example, "1-4-16"means that there are a decision node in the first period, four nodes in the second period and 16 nodes in the third period.) Method 2 is independent of the numbers of decision nodes, but we show the LPM of the terminal pension asset value calculated by Method 2 in the graph for comparison easily.

The smaller the LPM of terminal pension asset value is, the larger the number of decision nodes is. We can explain that LPM is reduced since the investment strategy in the discrete model becomes more flexible by increasing decision nodes. The LPM of pension asset value in Method 3 is smaller than the LPM in Method 2 because the LPM in Method 3 is calculated using the optimal strategy derived by minimizing LPM in the optimization model.

Figure 6 shows the distributions of terminal pension asset value for five kinds of decision nodes in Method 3. The right-hand side of Figure 3 is magnified view below 40% of cumulative probability in the left-hand side. As the number of decision nodes increases, the distribution of terminal pension asset value is approaching to the stepwise distribution. The shape of distribution below the target for pension asset value $W_G = 500$ shifts to the lower right, and the shape of distribution over the target shifts to the upper left.

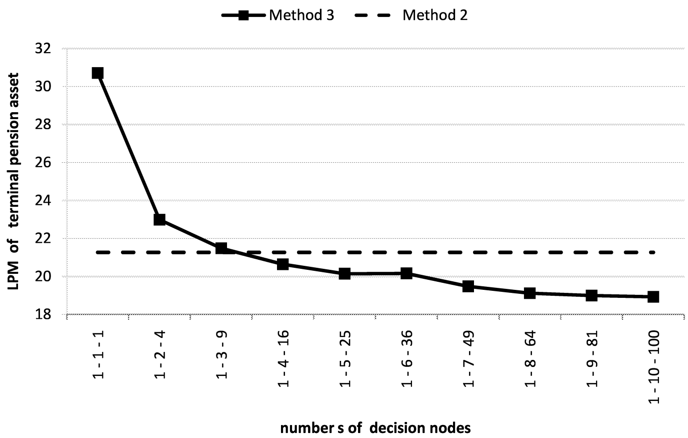

Figure 5: Difference caused by state discretization : LPM of terminal pension asset value

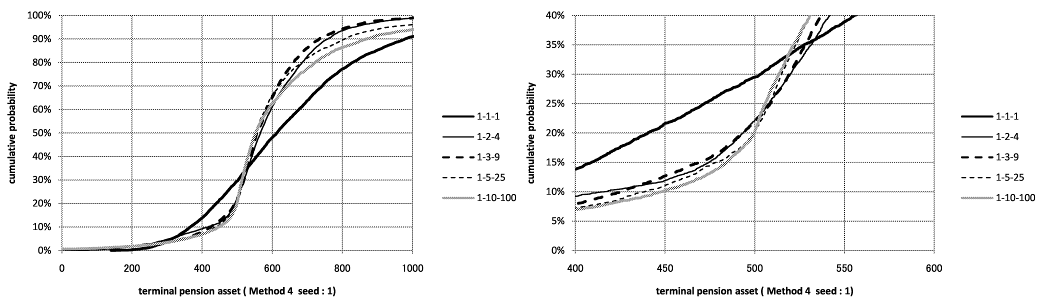

Figure 6: distribution of terminal pension asset value in Method 3

3.3 Difference between two models

We compare the optimal investment ratios in risky asset given by the continuous model with those of the discrete model at each time periods in Figure 7. The left-hand side of Figure 7 shows the ratios at the beginning of the first period($t = 0$), the middle shows the ratios at the beginning of the second period($t = 10/3$), and the right-hand side shows the ratios at the beginning of the third period($t = 20/3$). The top of Figure 7 shows the case of the fewest number of nodes, and the bottom shows the case of the most number of nodes. The horizontal axis shows the pension asset value. We plot points in Method 3, using the average pension asset values on the paths bundled through each node. For example, we plot a point at $t = 0$, two points at $t = 10/3$, and four points at $t = 20/3$ for '1-2-4' decision node in the second three graphs from the top.

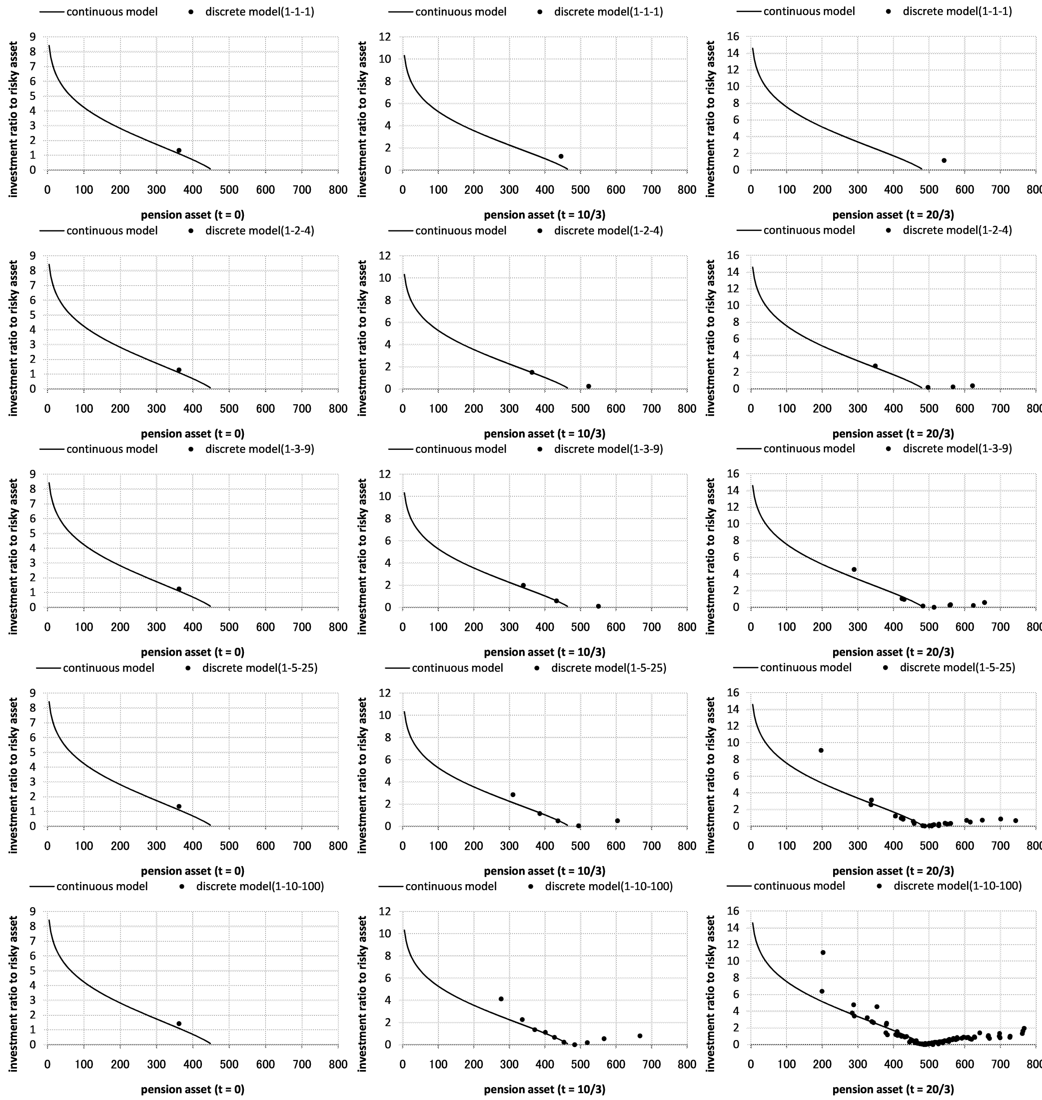

Figure 7: Investment ratio to risky asset given by continuous model and discrete model

The discrete model has almost the same optimal investment ratios as the continous model below 100% saving ratio. But the optimal strategy of the dicrete model tends to be riskier than that of continuous model for small pension asset value [14] .

[14] If the saving ratio is greater than 100%, we do not have to take risk and therefore we invest in cash and not in a risky asset. We can use the conservative investment strategy if the saving ratio runs to 100% before time horizon. The result shows the optimal strategy over 100% saving ratio is not to invest in a risky asset in the continuous model, but it is to invest in a risky asset in the discrete model. This is caused by the objective function of the

We show the investment ratios at $t = 0$ for '1-5-25' node which are calculated with five different sets of random numbers for ten kinds of initial pension asset values in Figure 8 in order to examine whether we have a stable tendency. We also find the investment ratios to risky asset in the discrete model are higher than in the continuous model. The less pension asset value is, the larger difference of the investment ratios between in the discrete model and in the continuous model are. This tendency exists in the case of different initial pension asset values even though we change a set of random numbers. This result shows this tendency is not caused only by a set of random numbers.

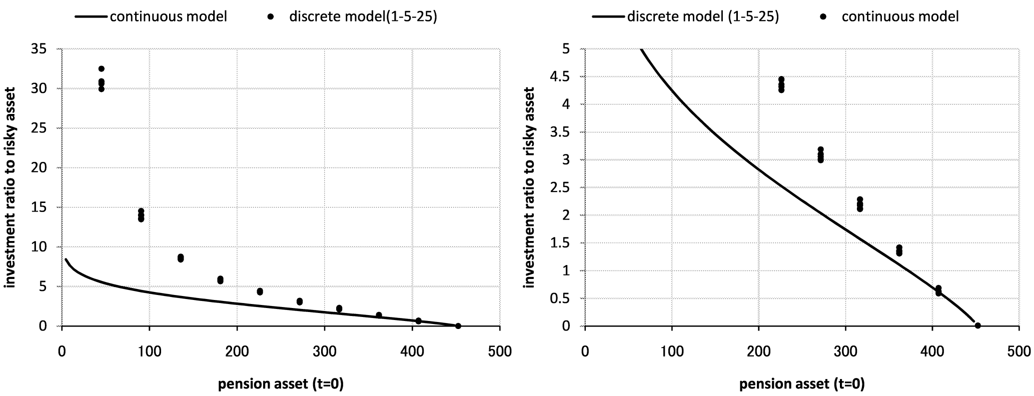

Figure 8: Investment ratio to risky asset at time 0

Difference of between the continuous model and the discrete model is caused by time and state discretization in general. However, we do not find the difference among five kinds of nodes in Figure 7. Therefore, the difference of the optimal investment ratio to risky asset between in the discrete model and in the continuous model is expected to be caused mainly by time discretization, not state discretization. We can explain the reason as follows. The pension asset value in Method 2 is lower than the value in Method 1 in Section 3.1. The investment ratio used in Method 2 is not optimal in the discrete model. The optimal investment ratio in the discrete model is expected to become larger because of increasing more pension asset value than the value in Method 2 by optimization.

4 Concluding remarks

This paper compares a continuous model and a discrete model for pension asset and liability management. There are three discretizations between the continuous model and the discrete model. These discretizations affect to the optimal strategy and the distribution of terminal pension asset value.

First, time discretization reduces the probability that the pension asset value exceeds the pension liability value. The continuous rebalancing earns more profit than discrete rebalancing, because the optimal strategy to minimize LPM is the contrarian strategy under the assumption in this paper. Next, the solution (optimal strategy and terminal pension asset value) in the discrete model is close to in the continuous model as number of the decision nodes increases. Random numbers affect the sampling error of the optimal strategy when the numbers of the state increase. Finally, the possibility of dependence on random numbers is caused discretization of the distribution of risky asset. When we build the multi-period stochastic programming model, we need to decide the number of state discretization thinking this trade-off.

We discuss the influences of two discretizaions using numerical calculation because it is difficult to divide these influences between state-discretization and discretization of the distribution of risky

discrete model in the numerical test. We include the term of maximizing average terminal pension asset value in the objective function because we want to derive the unique optimal solution of the discrete model though we set the weight $\gamma = 10^{-5}$ in the numerical tests to block the effect of this term. We adopt this weight to avoid numerical error, but we should have used smaller weight. This is our future research.

asset. As a future problem, we need to examine the influence of three discretizations analytically and numerically.

Appendix

A Verification of the optimal strategy given by discrete model

The optimal strategy of discrete model may depend on random numbers that were used to obtain it. We check that the optimal solution given by discrete model does not have strong dependence. We simulate investment strategy using the same as method that is used to simulate continuous optimal strategy, i.e. equation (12). However, we have to set the investment ratio to risky asset $\pi^{(i)}_{t_{k-1}}$ in equation (12). The simplest rule is that paths are bundled by pension asset value disregarding the difference of random numbers. Because the result (for example, pension asset value at time T) of simulation and of optimization must coincide when we use many random numbers enough.

Another rule is to interpolate the optimal strategy by the given optimal pension asset value. When the pairs of average of optimal pension asset value and optimal investment ratio to risky asset at time t and decision node $s = 1, \dots, S_t$ are $(\tilde{W}^s(t), \tilde{\rho}^s(t))$, If simulated pension asset value is $W(t)$, investment ratio to risky asset $\rho(t)$ is interpolated,

$$\rho(t) = \begin{cases} \tilde{\rho}^1(t) & W(t) \le \tilde{W}^1(t), \\ \tilde{\rho}^{s-1}(t) + \frac{W(t)-\tilde{W}^{s-1}(t)}{\tilde{W}^s(t)-\tilde{W}^{s-1}(t)} \left(\tilde{\rho}^s(t) - \tilde{\rho}^{s-1}(t)\right) & \tilde{W}^{s-1}(t) < W(t) \le \tilde{W}^s(t) \\ & \qquad (s = 2, \cdots, S_t), \\ \tilde{\rho}^{S_t}(t) & \tilde{W}^{S_t}(t) < W(t). \end{cases} \tag{13}$$

Where $\tilde{W}^s(t)$ are sorted disregarding state s, i.e. $\tilde{W}^1(t) \le \tilde{W}^2(t) \le \cdots \le \tilde{W}^{S_t}(t)$. The investment ratio to risky asset $\rho(t)$ given by this rule is not monotone in regard to pension asset value, because the optimal solution $(\tilde{W}^s(t), \tilde{\rho}^s(t))$ are uneven by random numbers or how to divide into states.

Figure 9 shows LPMs of terminal pension asset value optimized using discrete model (case A) and simulated using equation (12) and the optimized strategy given by discrete model (case B-1 ∼B-5). The left-hand side of Figure 9 shows the result derived by simple rule, the right-hand side of Figure 9 shows the result derived by interpolated optimal investment ratios. In the case of simulation (case B-1∼B-5), 5 results are simulated by different random seeds respectively. Case A and case B-1 are calculated the same random numbers.

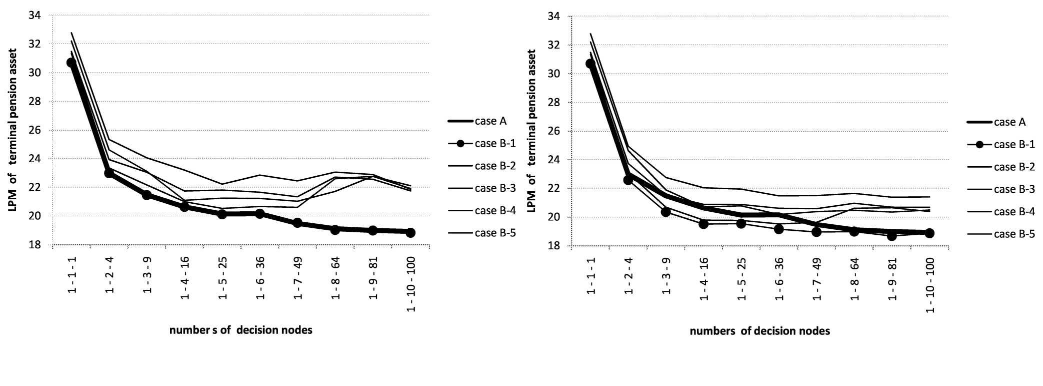

Figure 9: Verification of optimal strategy given by discrete model

Next, we can say 2 notions from the left-hand side of Figure 9. First, heavy line(case A) and marked line(case B-1) which their random seed are same are overlapping each other. The optimal strategy may include the error in the algorithm of case A because the investment ratio is calculated by pension asset calculated beforehand. As a result, this error is very small since the solutions of case A and case B-1 are close or same.

Second, the results of between case B-2∼B-4 which calculated using different random seeds each other are plotted as 4 thin lines. Because the LPM is increasing by replacement of their random seed, the optimal strategy depends on their random seed. There is more number of decision nodes, such difference is larger. Therefore the dependency to random numbers is higher as number of decision nodes is increasing.

We confirm the differece of LPM in case A from case B-2∼B-5 is reduced by interpolation by comparing the right-hand side with the left-hand side of Figure 9. We can reduce the dependence of random numbers by considering the discrete optimal strategy to be continuous.

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