

Optimal Symmetric No-trade Ranges in Asset Rebalancing Strategy with Transaction Costs

— An Application to the Government Pension Investment Fund in Japan —

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(Asia-Pacific Journal of Risk and Insurance, Published Online: 06/24/2014)

Abstract There are many studies of optimal asset allocation with transaction costs in academic literatures. However, those are numerically solved for two-asset and three-asset cases. In contrast, the investment in five asset classes (domestic bond and stock, international bond and stock, and cash) is at least required for practical pension fund management. Therefore, there are some real limitations to the continuous-time approach used in the previous literatures, which are the methods of solving the HJB equation for the stochastic control problem. In general, most investors use the rebalance rule with the no-trade ranges in practice which are constant and symmetric to the policy asset mix because it is easy to use the rule. In this paper, we propose the optimization model for the multiple asset allocation problem with transaction costs to determine the symmetric no-trade ranges of the policy asset mix using the derivative free optimization(DFO) approach proposed by Hibiki *et al.*(2014). Specifically, we solve the five-asset problem with boundary constraints for cash for the Government Pension Investment Fund(GPIF) in Japan in a discrete-time and finite-period setting. We clarify the fact that we need to adjust the amounts of risky assets even within the no-trade range if the boundary constraints for cash are required, and we describe the simulation procedure in the discrete-time model. We examine the difference for various time intervals and horizons, and conduct the sensitivity analysis for the various proportional transaction cost rates, the tracking error aversions, and the bounds for cash constraints. In addition, we compare the optimal time-dependent no-trade ranges with the constant no-trade ranges. The numerical results show the possibilities of applying the DFO model to the practical problem determining the symmetric no-trade ranges.

keyword: asset allocation, transaction cost, no-trade range, derivative free optimization

1. Introduction

The Government Pension Investment Fund(GPIF) in Japan needs to manage its fund based on the target allocation of policy asset mix, which is computed using mean values of the rates of returns of each asset, standard deviations of each asset and correlations among assets. Almost corporate pension funds also manage their funds in the same way, though their policy asset mixes are different from each other depending on the degree of maturing, guaranteed interest rate, and so on.

The actual weights of assets diverge from the target weights in the portfolio as the asset prices change randomly. Meanwhile, the transaction costs need to be paid to maintain the target weights by rebalancing assets. Therefore the investors do not rebalance their assets without exceeding the boundary of the no-trade region because it exists the trade-off between the costs associated with tracking error(deviations from the target allocation) and the transaction costs. As an example, the target allocation of policy asset mix and the

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permissible ranges of deviation from the target of the GPIF are shown in Table 1¹ .

Table 1: Policy asset mix of the GPIF

	Domestic bonds	Domestic stocks	International bonds	International stocks	Short-term assets
Target allocation	67%	11%	8%	9%	5%
Permissible range	±8%	±6%	±5%	±5%	—

The asset weights of the target portfolio in almost all pension funds are usually calculated by using the mean-variance model as well as the GPIF. However, it is unknown for the pension funds how to decide the no-trade ranges of their portfolios. The GPIF does not provide how to determine the permissible ranges and the theoretical rationale.

There are many academic studies for determining the boundaries of no-trade region and the optimal asset rebalancing strategy with transaction costs. Leland(1999) formulates the stochastic control problem in a continuous-time setting, and derives the optimal rebalancing strategy for an investor who minimizes both proportional transaction costs and tracking error with respect to a specified target asset mix for the three-asset case. Donohue and Yip(2003) conduct the sensitivity analysis of the modeling parameters associated with Leland(1999), and compare the different rebalancing strategies. Pliska and Suzuki(2004) improve the Leland model(1999), and derive the optimal trading strategy for an investor who minimizes both the fixed and proportional transaction costs and the tracking errors. Impulse control theory is used in a continuous-time and dynamic setting to deal with the problem in a general and analytical way, and the explicit solution for the two-asset case is derived.

Moreover, there are some continuous-time and discrete-time models in finite cases with the maximization of the expected CRRA utility of wealth. In a continuous-time approach, Liu and Loewenstein(2002) examine the optimal trading strategy for a CRRA investor who trades two assets(bond and stock), and maximizes the expected utility of wealth on a finite date with transaction costs. The closed-form solutions are obtained at an uncertain time by solving the Hamilton-Bellman-Jacobi(HJB) equation. Liu(2004) derives the optimal intertemporal consumption and investment policy of a CARA investor who faces the fixed and proportional transaction costs by solving the HJB equation. The sensitivity of the boundaries to the changes in risk aversion and volatility are shown as some further analysis of the optimal trading strategy. Lynch and Tan(2010) numerically solve the decision problem of a multi-period CRRA investor who faces fixed and proportional transaction costs and has access to two risky assets, both with predictable returns. Atkinson and Ingpochai(2010) examine the intertemporal optimal portfolio selection and the consumption rule of a CRRA investor who faces the proportional transaction costs when trading between a risk-free asset and multiple risky assets. Numerical examples for a portfolio with a risk-free asset and two risky assets are provided for constant variance as well as stochastic variance. Using a discrete-time approach, Gennotte and Jung(1994) solve the problem with a risky asset

¹The GPIF changed the midterm plan on June 7th, 2013. The standard deviations and the correlation coefficients of the rates of returns of assets are recalculated and replaced while the expected rates of returns of assets are unchanged. As the results, the policy asset mix is modified as follows — 60% for domestic bond, 12% for domestic stock, 11% for international bond, 12% for international stock, and 5% for cash(unchanged). The permissible ranges are not changed. We use the values of Table 1 because our study is done before the change of the policy asset mix.

and a riskless asset. They derive the optimal trading strategies described in terms of no transaction region using the binomial model. Boyle and Lin(1997) extend the work by Gennotte and Jung(1994), and derive an explicit closed-form solution for an investor who has a power utility function.

Some different types of the problems are formulated for multiple-asset cases, but only two-asset or three-asset problems are solved numerically. However, almost all investors of pension funds and investment trusts invest in more than five assets, such as domestic bond and stock, international bond and stock, and short-term asset(cash). There are some real limitations to solve the problem for the five-asset case in a continuous-time model by the HJB equation. In a real world, it is difficult to employ the utility functions assumed in the previous studies of finite horizon models. In addition, most institutional investors determine and manage the policy asset mix. We need to formulate a discrete-time model because we cannot trade assets continuously. In this paper, we formulate the finite and discrete-time model in the Leland approach which involves the transaction costs and the tracking errors in the objective function as well as Hibiki *et al.*(2014). We solve the problem with derivative free optimization(DFO), which is one of the mathematical programming approach. We can describe the simulation procedure flexibly in the DFO approach.

Hibiki *et al.*(2014) derive the no-trade region with the DFO approach for the Leland model(1999) in a discrete time and finite horizon setting, and show the usefulness of this approach. The optimal no-trade boundaries in a finite case are time-dependent. On the other hand, most investors use the rebalance rule with the no-trade ranges in practice which are constant and symmetric to the policy asset mix because it is easy to use the rule. In this paper, we propose the approach to derive the constant and symmetric no-trade ranges of the target portfolio of multiple assets numerically as with the no-trade ranges of the GPIF. In addition, we propose the model with the time-dependent no-trade ranges, and thus we compare them with the constant ranges.

We have two original contributions in this paper as follows.

- (1) Deriving the symmetric no-trade ranges for the multiple-asset problem, specifically with the GPIF parameters

There are no models to derive the symmetric no-trade ranges with respect to the policy asset mix for multiple-asset case used in practice because the practical use is not considered in the previous literatures. In this paper, we solve the five-asset problem with the GPIF parameters, and derive the optimal solutions. Our contribution is to compute the practical solutions(symmetrical no-trade ranges or rectangles) for four risky asset-case rather than the theoretical solutions(parallelogram-like no-trade regions) for two risky asset-case ².

- (2) Adjustment algorithm for the rebalancing strategy

We clarify the fact that we need to adjust the amounts of risky assets even within the no-trade ranges when the boundary constraints for cash are required. We describe the simulation procedure involving the adjustment algorithm in the discrete-time model.

This paper is organized as follows. In Section 2, we clarify the fact that we need to adjust the amounts of risky assets even within the no-trade ranges, and we describe the simulation

²Many previous studies show that the optimal no-trade region is non-symmetric with respect to the policy asset mix or parallelogram-like policy(See Leland(1999), Liu(2004), Lynch and Tan(2010), Hibiki *et al.*(2014)). We can compute the no-trade region and implement the rebalance strategy for two risky asset-case. But four risky assets are usually invested for pension fund management, and therefore we use the symmetric no-trade ranges from a practical perspective. However, it is important to compare the results between the symmetric no-trade range(rectangular policy) and general no-trade region(parallelogram-like policy). We show the comparison for two risky asset-case in Appendix A.

procedure in the discrete-time model. We solve the optimization problem to decide the no-trade(permissible) ranges with respect to the policy asset mix using the parameters applied for the GPIF, and examine them in Section 3. In Section 4, we derive the optimal time-dependent no-trade ranges, and compare them with the constant ranges. Section 5 provides our concluding remarks.

2. Determination of the no-trade ranges and DFO modeling

In this paper, we solve the problem to decide the no-trade ranges with the DFO method as well as Hibiki *et al.*(2014). The problem with the proportional transaction costs are solved minimizing the sum of the transaction costs and the tracking errors associated with the target allocation as well as Leland(1999)³. In this paper, the rebalance targets are set to the no-trade boundaries of ranges when assets exceed them because the problems with the proportional transaction costs are solved under the same setting in the previous papers including Leland(1999)⁴. The DFO method is the non-linear optimization method where the problems are solved without the derivative of the objective function⁵. The optimal solution can be obtained precisely for the problem where the minimized objective function is convex, and the number of decision variables is several. The number of decision variables is $2^{N-1}(N-1)$ in the N -asset case(including a riskless asset) of the Leland model(1999), and increases exponentially with N . However, the no-trade ranges of the GPIF in Table 1 are symmetric to the target allocation, and therefore the number of decision variables is $N-1$ which is the same as the number of risky assets. The type of the problem goes well with the DFO method.

We describe the simulation procedure of calculating the objective function value to derive the optimal solution by the DFO method. We discretize the stochastic process for the risky assets used in Leland(1999), and generate the normal random number for ε_i by using the Monte Carlo method.

$$\frac{\Delta S_i}{S_i} = \mu_i \Delta t + \sigma_i \sqrt{\Delta t} \varepsilon_i, \quad \varepsilon_i \sim N(0, 1), \quad (2.1)$$

$$\text{correl}(\varepsilon_i, \varepsilon_j) = \rho_{ij}, \quad (2.2)$$

where S_i is the price of asset i , μ_i is the expected rate of return of asset i , σ_i is the standard deviation of asset i , and ρ_{ij} is the correlation coefficient between asset i and j . We calculate the weights of risky assets from the prices, and rebalance the assets based on the rule. We formulate the model to derive the no-trade ranges with respect to the given policy asset mix by minimizing the sum of the transaction costs and the tracking errors. We do not involve the no-trade range of asset N , because we impose the boundary constraints for cash where the lower bound of the asset N is L_N , and the upper bound is U_N ⁶.

³We show the rationale for using the trade-off between tracking error and transaction cost in the no-trade range problem in accordance with Leland(1999) in Appendix B.

⁴The no-trade boundary is the optimal rebalance target in the infinite and continuous-time model with “proportional” transaction costs as in Leland(1999), but we cannot guarantee it in the finite and discrete-time model. However, Hibiki *et al.*(2014) showed that the boundary value of the no-trade range is still appropriate as the rebalance target for two-asset case with a risky asset and a riskless asset, and therefore we define the problem based on the rebalance strategy from a practical perspective.

⁵The readers can refer Conn, Scheinberg and Vicente(2009) and the chapter nine of Nocedal and Wright(2006) in detail. In this paper, we use NUOPT/DFO(2011) added on the mathematical programming software package called NUOPT developed by NTT DATA Mathematical Systems, Inc. NUOPT/DFO implements the trust-region methods based on derivative free models, which maintain quadratic models based only on the objective function values computed at sample points.

⁶In the case of the GPIF as in Table 1, the asset N is short-term asset or cash. We need to have cash

2.1. Imposing the boundary constraints for cash

When we rebalance a risky asset out of the no-trade range to the boundary, there are some cases that we need to adjust the amount of other risky assets even within the no-trade ranges⁷. An example is shown in Table 2. Suppose we have a target portfolio at time 0, and set up $L_N = 3\%$. Assume that the time passed and asset prices changed. At time t , we assume that the weight of the domestic bond becomes 53%, and the weight of the domestic stock becomes 19%. Because both assets are beyond the no-trade ranges, the weight of the domestic bond is moved from 53% to 59% which is the lower no-trade boundary, and the weight of the domestic stock is moved from 19% to 17% which is the upper no-trade boundary. In contrast, we do not need to rebalance the international bond and stock if we follow the rebalance rule because both assets are within the no-trade ranges. As the result, the calculated weight of cash is -3% , but we need to have 3% cash. Therefore, we have to decrease 6% of the risky assets at least to increase 6% cash. This occurs because the asset weight substantially below the lower no-trade boundary needs to be increased. If we do not prevent this situation by decreasing other assets which are over the target weights, the total weight of the assets except asset N becomes beyond $1 - L_N$. In the case of Table 2, all of risky assets can be within the no-trade ranges by decreasing 2% of the assets even within the no-trade ranges respectively. This is implemented by the reallocation from "Rebalance (1)" to "Rebalance (2)". We involve the adjustment algorithm with the above-mentioned steps to decide the no-trade ranges as follows⁸.

Table 2: Example of adjustment steps within the no-trade ranges

Asset class	Domestic bond	Domestic stock	International bond	International stock	Short-term asset
Range	59 ~ 75%	5 ~ 17%	3 ~ 13%	4 ~ 14%	—
time 0	67%	11%	8%	9%	5%
⋮	⋮	⋮	⋮	⋮	⋮
time t	53%	19%	13%	14%	1%
Rebalance (1)	59%	17%	13%	14%	-3%
Rebalance (2)	59%	15%	11%	12%	3%

- (1) We calculate the excess weight EW which needs to be adjusted.
- (2) We examine the set I of assets above the target weights of policy asset mix, and calculate EX_i which denotes the difference between the investment weight of asset i and the target weight. One asset is at least above the target weight.
- (3) We solve the problem to minimize the sum of transaction costs and tracking errors at each time and on each path in order to obtain the optimal decreasing weight of asset i

in a degree in order to pay for pension money, while we need to invest in risky assets without having much cash because of the efficient investment. When we do not impose the boundary constraints, we have $L_N = 0$ and $U_N = 1$.

⁷We do not need to adjust the risky assets within the no-trade ranges in the continuous-time model because even the asset out of the no-trade range is rebalanced on the boundary immediately. However, we need to adjust the risky assets within the no-trade ranges even in the continuous-time model if the lower bound of asset N is greater than zero, that is $L_N > 0$.

⁸If the no-trade ranges are small, we do not need to adjust the amounts of assets, however we need to pay much transaction costs. Therefore, the optimal no-trade ranges are derived in consideration of the trade-off.

or Δw_i .

$$\sum_{i \in I} \Delta w_i = EW, \quad 0 \leq \Delta w_i \leq EX_i$$

When the cash is beyond the upper limit, the amounts of risky assets are adjusted based on the similar ways. We omit the algorithm because of the similar steps.

2.2. Simulation procedure

Let q_i ($i = 1, \dots, N - 1$) denote the no-trade range of asset i . We describe the simulation procedure and solve the optimization problem by the DFO method in what follows⁹. The superscript of ‘ b ’ shows the notation of pre-rebalance, and the superscript of ‘ a ’ shows the notation of post-rebalance. We let p_i the target weight of asset i of the policy asset mix.

- (1) Suppose that the amount of wealth at time 0 is 1 or $W_0^{(m)} = 1$, and the value of asset i is p_i , or $S_{i0}^{a(m)} = p_i$ ¹⁰.
- (2) We calculate sequentially the value of asset i ($S_{it}^{b(m)}$), the weight ($w_{it}^{b(m)}$) and the amount of wealth ($W_t^{(m)}$) on path m at time t ($t = 1, \dots, T$; $m = 1, \dots, M$; $i = 1, \dots, N$), where M is the number of random samples, and T is the number of periods. We generate random samples $\varepsilon_{it}^{(m)}$ used in Equation (2.3) by the Monte Carlo method. $S_{i,t-1}^{a(m)}$ is computed in Equation (2.19) to be described later.

$$S_{it}^{b(m)} = \left(1 + \mu_i \Delta t + \sigma_i \sqrt{\Delta t} \varepsilon_{it}^{(m)}\right) S_{i,t-1}^{a(m)} \quad (2.3)$$

$$W_t^{(m)} = \sum_{i=1}^N S_{it}^{b(m)} \quad (2.4)$$

$$w_{it}^{b(m)} = \frac{S_{it}^{b(m)}}{W_t^{(m)}} \quad (2.5)$$

We compute the weight of asset i after rebalancing ($w_{it}^{a(m)}$) as in Equations (2.6) and (2.7) based on the value of $w_{it}^{b(m)}$,

$$w_{it}^{a(m)} = \begin{cases} p_i - q_i & \text{for } w_{it}^{b(m)} \in [0, p_i - q_i), \\ w_{it}^{b(m)} & \text{for } w_{it}^{b(m)} \in [p_i - q_i, p_i + q_i], \\ p_i + q_i & \text{for } w_{it}^{b(m)} \in (p_i + q_i, 1], \end{cases} \quad (i \in I_{N-1}) \quad (2.6)$$

$$w_{Nt}^{a(m)} = 1 - \sum_{i \in I_{N-1}} w_{it}^{a(m)}, \quad (2.7)$$

where $I_{N-1} = \{1, 2, \dots, N - 1\}$ denotes the set of assets except asset N . As discussed in Section 2.1, if the weight of asset N ($w_{Nt}^{a(m)}$) is beyond the upper or lower limit when rebalancing the assets except asset N on the boundary of the no-trade range, we need to adjust the weights of the assets except asset N in what follows.

- ① The case that the weight of asset N is below the lower limit, or $w_{Nt}^{a(m)} < L_N$

We calculate the excess weight, or $EW_t^{(m)} = L_N - w_{Nt}^{a(m)}$, and adjust the asset weights as follows.

⁹Our procedure may be complicated. The reason is we impose the constraints (or upper and lower limits) for cash. However, according to the numerical analysis (the left graph of Figure 6) in Section 3.1, the probabilities of which cash is adjusted due to the constraints are at most about 6% for $\Delta t = 0.1$ year and about 14% for $\Delta t = 1$.

¹⁰The investment value is not dependent on path m at time 0, but we describe the notation for convenience.

- a. Let $I_t^{+(m)}$ denote the set of assets which meets the condition that $w_{it}^{a(m)} > p_i$ ($i \in I_{N-1}$).
- b. We solve the following subproblem of minimizing the objective function to obtain the values of $\Delta w_{it}^{(m)}$ which are the amounts of the decreases in the asset weights in the set $I_t^{+(m)}$ ¹¹.

$$\text{Minimize } C(\Delta \mathbf{w}_t^{(m)}) - C(\mathbf{0}), \quad (2.8)$$

$$\text{subject to } 0 \leq \Delta w_{it}^{(m)} \leq EX_{it}^{(m)} \quad (i \in I_t^{+(m)}), \quad (2.9)$$

$$\sum_{i \in I_t^{+(m)}} \Delta w_{it}^{(m)} = EW_t^{(m)}, \quad (2.10)$$

$$\Delta w_{it}^{(m)} = 0 \quad (i \in I_{N-1} - I_t^{+(m)}), \quad (2.11)$$

$$\Delta w_{Nt}^{(m)} = -EW_t^{(m)}, \quad (2.12)$$

where $\Delta \mathbf{w}_t^{(m)} = (\Delta w_{1t}^{(m)}, \dots, \Delta w_{Nt}^{(m)})$, $EX_{it}^{(m)}$ is the investment weight minus the target weight, or $EX_{it}^{(m)} = w_{it}^{a(m)} - p_i$, and $\mathbf{0}$ is the N -dimensional zero vector. According to the after-mentioned Equation (2.20), $C(\mathbf{x})$ is the objective function on path m at time t . We can write it as follows,

$$C(\mathbf{x}) = \lambda \sum_{i=1}^N \sum_{j=1}^N \left(EX_{it}^{(m)} - x_i \right) \left(EX_{jt}^{(m)} - x_j \right) \sigma_{ij} + \sum_{i \in I_{N-1}} k_i x_i, \quad (2.13)$$

where λ is a tracking error aversion, and k_i is a proportional transaction cost rate of asset i .

- c. Let $\Delta w_{it}^{(m)*}$ denote the optimal solution of the subproblem, and we replace $w_{it}^{a(m)} - \Delta w_{it}^{(m)*}$ with $w_{it}^{a(m)}$ adjusted to avoid using the another notations.

② The case that the weight of asset N is over the upper limit, or $w_{Nt}^{a(m)} > U_N$

We calculate the deficit weight or $DW_t^{(m)} = w_{Nt}^{a(m)} - U_N$, and adjust the asset weights as follows.

- a. Let $I_t^{-(m)}$ denote the set of assets which meets the condition that $w_{it}^{a(m)} < p_i$ ($i \in I_{N-1}$).

- b. We solve the following subproblem of minimizing the objective function to obtain the values of $\Delta w_{it}^{(m)}$ which are the amounts of the increases in the asset weights in the set $I_t^{-(m)}$.

$$\text{Minimize } C(\Delta \mathbf{w}_t^{(m)}) - C(\mathbf{0}), \quad (2.14)$$

$$\text{subject to } 0 \leq \Delta w_{it}^{(m)} \leq DE_{it}^{(m)} \quad (i \in I_t^{-(m)}), \quad (2.15)$$

$$\sum_{i \in I_t^{-(m)}} \Delta w_{it}^{(m)} = DW_t^{(m)}, \quad (2.16)$$

$$\Delta w_{it}^{(m)} = 0 \quad (i \in I_{N-1} - I_t^{-(m)}), \quad (2.17)$$

$$\Delta w_{Nt}^{(m)} = -DW_t^{(m)}, \quad (2.18)$$

where $DE_{it}^{(m)}$ is the target weight minus the investment weight or $DE_{it}^{(m)} = p_i - w_{it}^{a(m)}$.

¹¹As in Equation (2.11), the decreases in the asset weights except the assets in the set $I_t^{+(m)}$ and asset N are equal to zero.

c. We replace $w_{it}^{a(m)} + \Delta w_{it}^{(m)*}$ with $w_{it}^{a(m)}$ adjusted.

③ The value of asset i after rebalancing ($S_{it}^{a(m)}$) is calculated in Equation (2.19)¹²,

$$S_{it}^{a(m)} = w_{it}^{a(m)} W_t^{(m)} \quad (i = 1, \dots, N). \quad (2.19)$$

(3) The model is formulated as follows,

$$\begin{aligned} \text{Minimize}_{q_i} \quad C \equiv & \frac{1}{M} \sum_{m=1}^M \sum_{t=1}^T e^{-(r\Delta t)t} \left\{ \lambda \sum_{i=1}^N \sum_{j=1}^N \left(w_{it}^{a(m)} - p_i \right) \left(w_{jt}^{a(m)} - p_j \right) \sigma_{ij} \right. \\ & \left. + \sum_{i=1}^{N-1} k_i \left| w_{it}^{a(m)} - w_{it}^{b(m)} \right| \right\}, \end{aligned} \quad (2.20)$$

$$\text{subject to} \quad 0 \leq q_i \leq \max(p_i, 1 - p_i) \quad (i = 1, \dots, N - 1), \quad (2.21)$$

$$L_N \leq w_{Nt}^{a(m)} \leq U_N \quad (t = 1, \dots, T; m = 1, \dots, M). \quad (2.22)$$

3. Application to the GPIF

The GPIF manages asset allocation on the basis of the policy asset mix for five assets as in Table 1; domestic bond(DB), domestic stock(DS), international bond(IB), international stock(IS), and short-term asset(SA). Different three cases of the appreciation rate for total factor productivity are shown in Ministry of Health, Labour and Welfare(2008)¹³. In this paper, we use the expected rates of returns estimated for the intermediate case. We examine the model using the expected values, the standard deviation and the correlation coefficients of the rates of returns shown in Table 3. The other parameters are as follows¹⁴.

- Proportional transaction cost rates
(DB) $k_1 = 0.2\%$, (DS) $k_2 = 1.2\%$, (IB) $k_3 = 0.2\%$, (IS) $k_4 = 1.2\%$
- Tracking error aversion : $\lambda = 3$

¹²The weight of asset N is

$$w_{Nt}^{a(m)} = \begin{cases} L_N & \text{for } EW_t^{(m)} > 0 \\ U_N & \text{for } DW_t^{(m)} > 0 \\ 1 - \sum_{i \in I_{N-1}} w_{it}^{a(m)} & \text{otherwise.} \end{cases}$$

¹³They calculate three kinds of estimates for the appreciation rates for total factor productivity; 1.3%, 1.0% and 0.7%. They estimate the different expected rates of returns, respectively.

¹⁴The transaction costs used by the GPIF are not published, and there are a few previous studies which describe the transaction costs. Sasaki(2006) sets 0.5% for stocks and 0.2% for bonds. Chida(2004) sets 1.2% for stocks and 0.2% for bonds as the transaction cost rates including market impact costs. In this paper, we use the rates in Chida(2004).

We set the tracking error aversion to three($\lambda = 3$) from the following reason. As shown in Appendix B, we may choose the same value as the risk aversion for the tracking error aversion in the case of the simple mean-variance problem. It is published that the policy asset mix is derived by solving the mean-variance optimization problem with the parameters in Table 3 under the constraints that the expected rate of return is equal to 4.1%. The reciprocal of the dual value for the constraint corresponds to the risk aversion of the objective function which consists of the expected value and the variance of the rate of return, and the risk aversion is 2.99. The mean-variance optimization problem for the GPIF is different from the problem shown in Appendix B because of adding some constraints. Unfortunately, we have no idea to determine the appropriate value of λ except the simple mean-variance problem. Therefore, we approximately assigned three($\lambda = 3$) to the tracking error aversion.

- Boundary constraints for short-term asset(cash constraints)
(lower bound) $L_N = 3\%$, (upper bound) $U_N = 10\%$
- Number of sample paths : $M = 10,000$

Table 3: Return, risk and correlation

	Domestic bond(DB)	Domestic stock(DS)	International bond(IB)	International stock(IS)	Short-term asset(SA)
Expected return	3.7%	6.0%	3.7%	6.2%	2.2%
Standard deviation	5.45%	22.25%	13.44%	19.85%	3.71%
Correlation	DB	DS	IB	IS	SA
Domestic bond	1.00	0.15	-0.06	-0.05	0.45
Domestic stock	0.15	1.00	-0.26	0.27	-0.01
International bond	-0.06	-0.26	1.00	0.55	-0.05
International stock	-0.05	0.27	0.55	1.00	-0.12
Short-term asset	0.45	-0.01	-0.05	-0.12	1.00

All of the problems are solved using NUOPT/DFO added on the mathematical programming software package called NUOPT(Ver. 14.1) developed by NTT DATA Mathematical Systems, Inc. – on Windows XP personal computer which has 2.40 GHz CPU and 3GB memory.

3.1. Numerical analysis

We examine the model for 28 combinations of four kinds of time intervals($\Delta t = 0.1, 0.2, 0.5, 1.0$) and seven kinds of time horizons(5, 10, 15, 20, 30, 40, 50 years)¹⁵. Figure 1 is shown to compare the results for different time horizons, and Figure 2 is shown to compare the results for different time intervals.

Figure 1 shows that the no-trade ranges converge to certain values as the time horizon becomes large. The shorter the time horizon is, the larger the no-trade ranges are on average because investors tend to adopt the no-trade strategy to cut the trading costs on the rebalancing point close to the maturity. Figure 2 shows that the no-trade ranges are reduced as the time interval is large. The investors reduce the no-trade range and avoid the increase in the tracking error calculated by the quadratic function because the investors cannot trade assets except the time points in the discrete-time model. These results can be also shown in the case of more than three assets as well as in Hibiki *et al.*(2014). The relationship among the no-trade ranges of four risky assets is

$$DB > DS > IS > IB.$$

It is the same as the no-trade(permissible) ranges of the GPIF.

¹⁵The number of periods is calculated by years divided by the time intervals. For example, the number of periods is 100 when $\Delta t = 0.1$ and 10 year horizon.

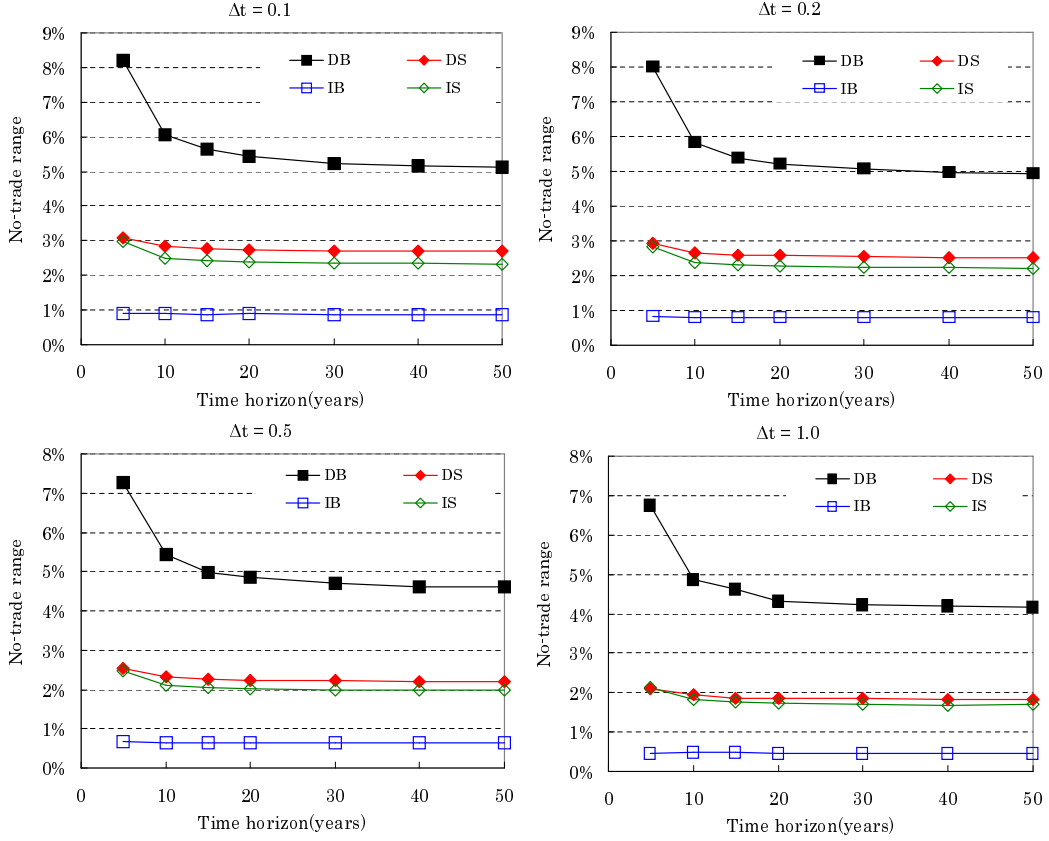


Figure 1: Comparison of the results for different time horizons

This shows that the policy asset mix affects their no-trade ranges¹⁶. We calculate the different no-trade ranges for their target portfolios and show Figure 3 to verify the reasonableness of the results. We derive ten kinds of target portfolios for their expected rates of returns using the mean-variance model with parameters in Table 3¹⁷. We show the no-trade ranges corresponding to the portfolios in the left-hand side of Figure 3, and the relationship between the portfolio weights and the no-trade ranges in the right-hand side of Figure 3. The expected rate of return of P1 is the lowest, and that of P10 is the highest in order. As the results, the weights of DB become small and the weights of DS and IS have large and similar values in proportion as the expected rates of returns of the portfolios become high. The weights of IB become small toward the P10 after achieving the peak at P3.

¹⁶Figure 7 shows that the no-trade ranges become wide as the proportional transaction cost rates increase. The no-trade ranges of the stocks become wide relatively to the bonds because the proportional transaction cost rates of the stocks are 1.2%, and those of the bonds are 0.2%. The no-trade range is more affected by the target ratio than the proportional transaction cost for the domestic bond. We conduct the sensitivity analysis for the proportional transaction cost in detail in Section 3.2.

¹⁷The GPIF uses the following constraints to derive the target portfolio; the target ratio of short-term asset(cash) = 5%, the expected rate of return of the target portfolio = 4.1%, and the relationship among the target weights of the assets is $DS > IS > IB$. In particular, we impose constraints on the above-mentioned relationship among asset weights as follows ; DS weight \geq IS weight + 1% and IS weight \geq IB weight + 1%. We solve the problem by the mean-variance model, and obtain the optimal weights of 66.7% for DB, 10.4% for DS, 8.4% for IB and 9.4% for IS. When we round the weights to 1% unit, the weights of the target portfolio in Table 1 are obtained. The policy portfolio is the P3 portfolio. The other portfolios(P1 to P10 except P3) can be also derived by solving the mean-variance problems with the expected rates of returns of 3.9%, 4.0%, 4.3%, 4.5%, 4.7%, 4.9%, 5.1%, 5.3% and 5.5%, respectively.

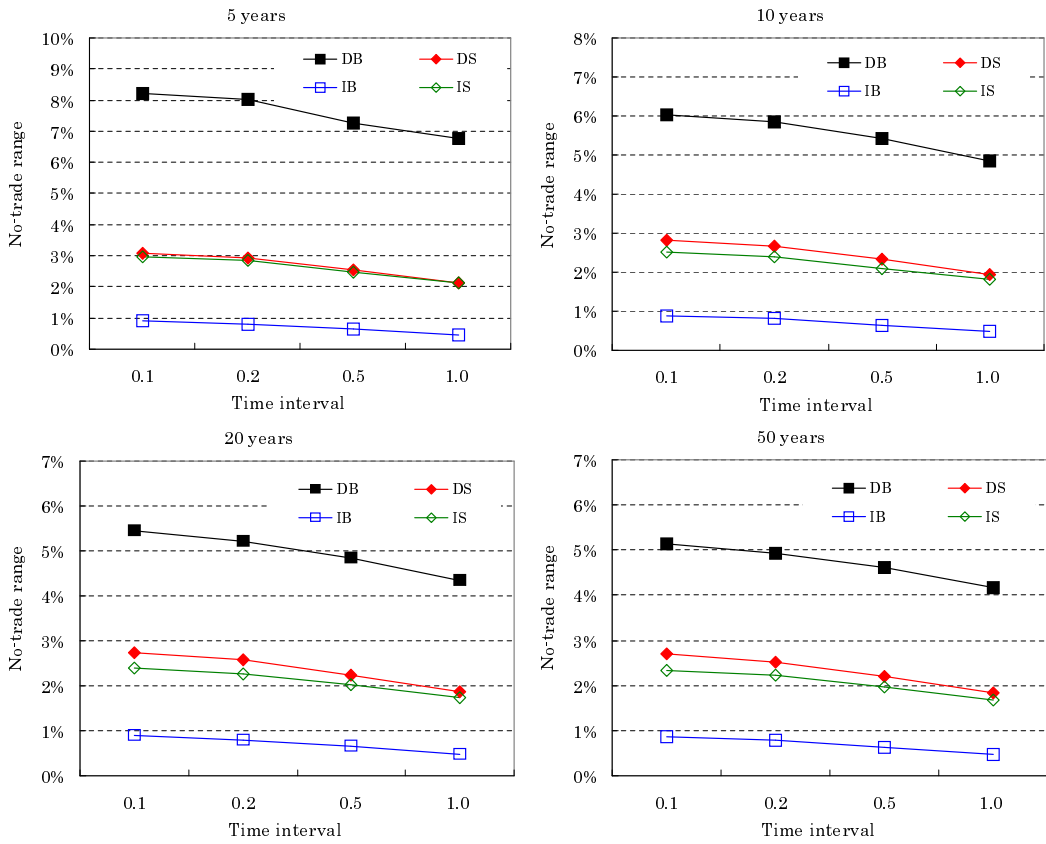


Figure 2: Comparison of the results for different time intervals

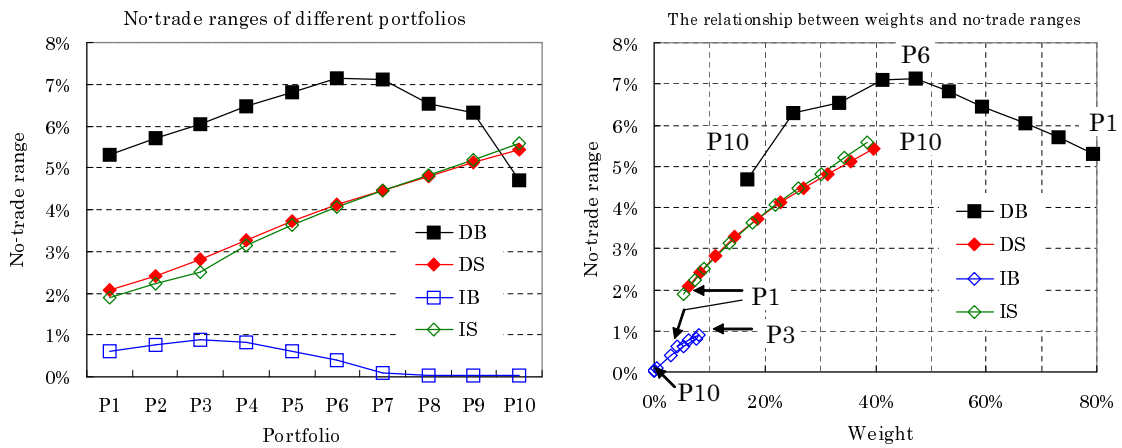


Figure 3: No-trade ranges for different target portfolios

The right-hand side of Figure 3 shows that the no-trade ranges increase as the weights approach 50%, and the no-trade ranges decrease as the weights lose touch with 50%. In particular, the no-trade ranges of DS are similar to those of IS, and they become large as the expected rates of returns of the portfolios increase because their weights are less than 50%. The no-trade ranges of DB are getting large for P1 to P6 portfolios, but they become small after achieving the peak at P6. The no-trade ranges of IB become small from P3 portfolio. Meanwhile, the no-trade ranges of IB are getting large for P1 to P3 portfolios, but they become small after achieving the peak at P3.

We examine the sensitivity of the optimal no-trade range of each asset to the objective function value. In particular, we calculate the corresponding objective function to the different no-trade ranges of the certain asset except which we fix the optimal no-trade ranges of other assets. We show the graphs in Figure 4 where the horizontal axis is the multiple of the optimal no-trade range, and the vertical axis is the multiple of the objective function value. The values of 1 on the both axes correspond to the optimal values. Figure 4 shows the results for the case of $\Delta t = 0.1$ and 5 year horizon on the left-hand side, and for the case of $\Delta t = 0.1$ and 10 year horizon on the right-hand side.

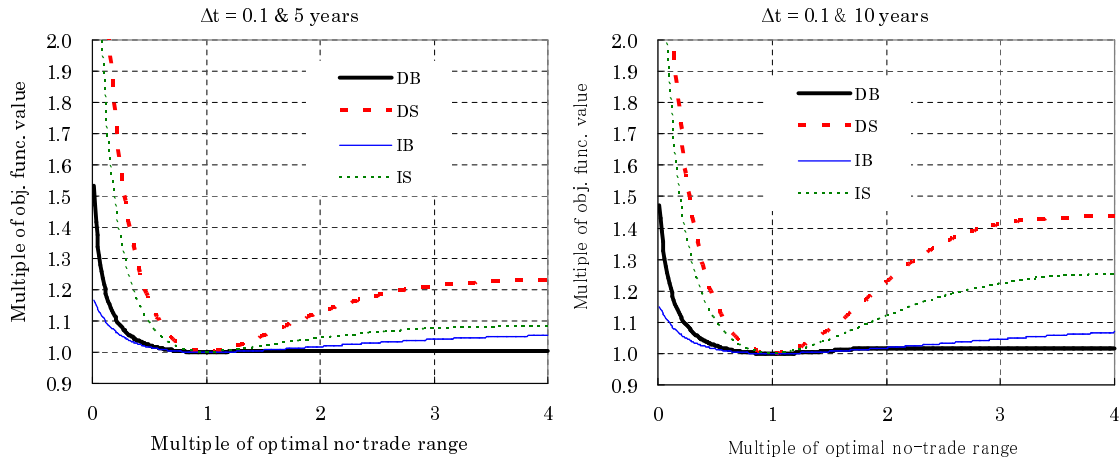


Figure 4: Objective function values for different no-trade ranges

The degrees of convexity of the objective function are DS, IS, IB, and DB in order. This order is the same as that of their volatilities. The increase in tracking error needs to be larger than the decrease in the transaction cost to obtain the convex objective function without degeneracy because the transaction costs approach zero as the no-trade ranges become large. The volatilities of the assets affect the tracking error, and therefore the order of the degrees of convexity is the same as that of the volatilities. As the no-trade ranges become large, the objective function is not degenerate, but the increase in the objective function values for IB and DB get smaller. This means that the sensitivity of the no-trade ranges to the objective function is small and we have the possibilities of solving the problems unstably. We need to pay attention to it when solving the problems.

Figure 5 shows the objective function values for the different no-trade ranges of DB and IB calculated under four kinds of years with $\Delta t = 0.1$, and four kinds of time intervals with 10 year horizon. The sensitivity of the objective functions for DB are similar around the optimal solution even the different number of years(time horizons) and the different time intervals. In addition, the objective function values are insensitive to the larger no-trade range of DB, and the curves begin to flatten at the smaller no-trade range(smaller multiple

of the horizontal axis in Figure 5) as the number of years and the time intervals are smaller. On the other hand, the curve of the objective function value for IB tends to be flattened as the time intervals are large and the time horizons are short.

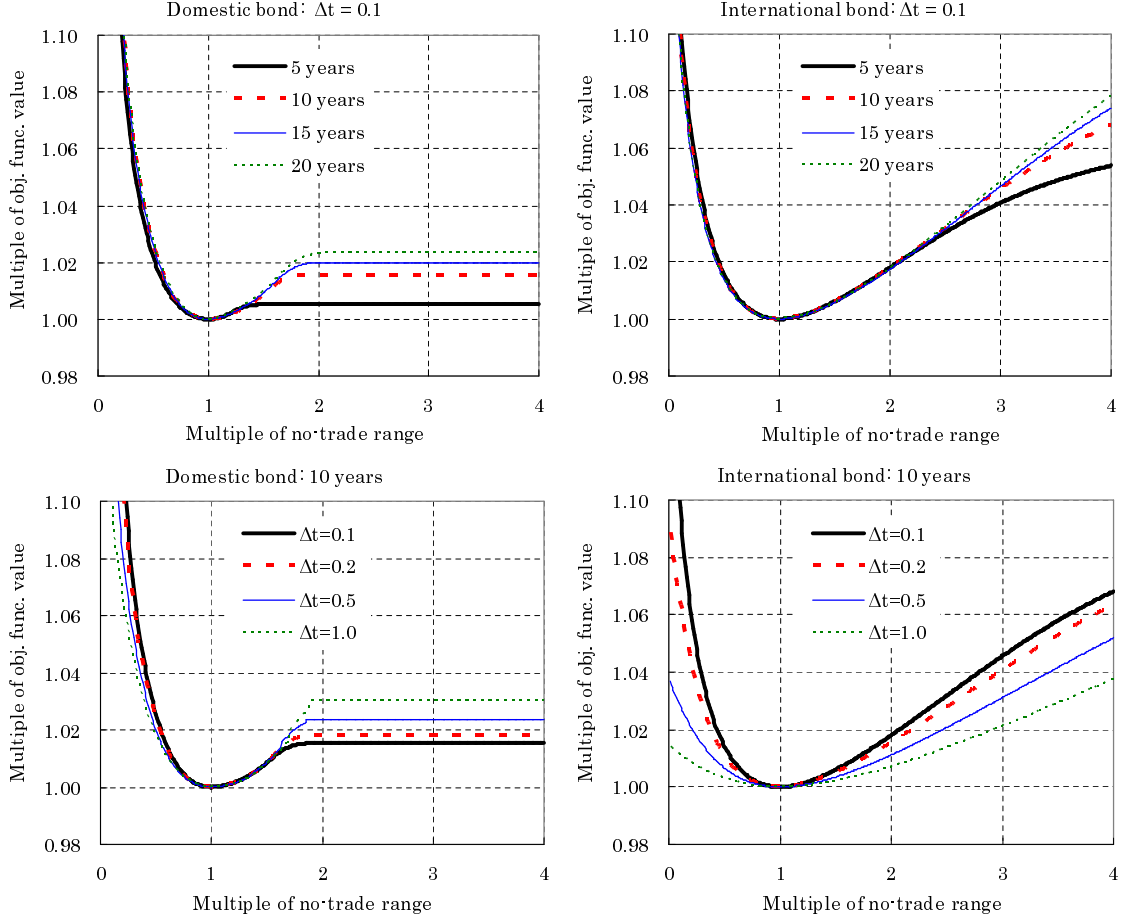


Figure 5: Objective values for different no-trade ranges of domestic and international bonds

We examine the probability that the rebalance is needed to impose the cash constraints even if the asset weights are within the no-trade ranges. We call it cash adjusted probability. We show the probability and the computation time in Figure 6. The cash adjusted probability is defined as the sum of the probabilities that the weight of asset N is below the lower bound L_N and above the upper bound U_N . It is calculated by Equation (3.1) using the excess weight $EW_t^{(m)}$ and the deficit weight $DW_t^{(m)}$ in the simulation procedure of Section 2.2,

$$\varphi = \frac{1}{T \cdot M} \sum_{t=1}^T \sum_{m=1}^M \left(\mathbf{1}_{\{EW_t^{(m)} > 0\}} + \mathbf{1}_{\{DW_t^{(m)} > 0\}} \right), \quad (3.1)$$

where $\mathbf{1}_{\{A\}}$ denotes the indicator function which value is 1 if the condition A is satisfied, or 0 otherwise. The cash adjusted probability becomes small for the small time interval or the long time horizon because the variability of the risky asset for a period is relatively small. The computation time increases for the smaller time interval and the longer time horizon because the number of periods increases.

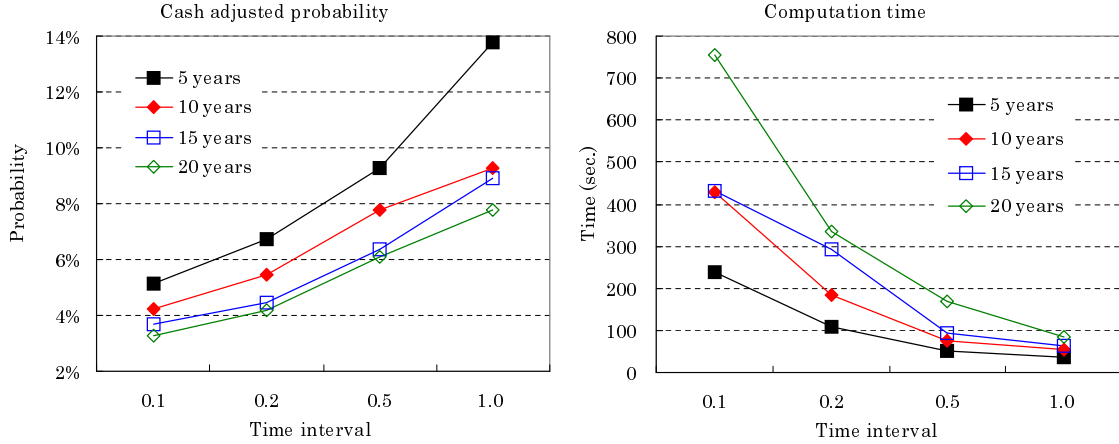


Figure 6: Cash adjusted probabilities and computation time

3.2. Sensitivity analysis

We conduct the sensitivity analysis for three kinds of parameters (proportional transaction cost rate of stocks k_2 and k_4 , tracking error aversion λ and cash constraint parameter α) as follows.

$$\begin{aligned}
 k_2 = k_4 &= 0.2\%, 0.6\%, 1.0\%, 1.2\%, 1.5\%, 2.0\% \\
 \lambda &= 1, 2, 3, 4, 5, 10, 20, 30 \\
 \alpha &= 0.0, 0.5, 1.0, 1.5, 2.0, 2.5
 \end{aligned}$$

We set the lower bound L_N and the upper bound U_N using the following equations,

$$\begin{aligned}
 L_N &= (1 - \alpha)p_N + \alpha L_N^0 \text{ for lower bound,} \\
 U_N &= (1 - \alpha)p_N + \alpha U_N^0 \text{ for upper bound,}
 \end{aligned}$$

where $L_N^0 = 3\%$, $U_N^0 = 10\%$, $p_N = 5\%$, $\Delta t = 0.1$ (year) and the time horizon is 10 years.

We show the results of the sensitivity analysis in Figures 7 to 9. The left sides include the no-trade ranges of the four risky assets and the right sides include the objective function values, transactions costs and tracking errors.

Figure 7 shows that the transaction costs increase in proportion to the cost rates. However, the no-trade ranges become large to avoid the trading assets, and tracking error increases simultaneously. Figure 8 shows that the no-trade ranges decrease to keep the tracking error small as the tracking error aversion increases. The no-trade ranges are small but consequently the transaction costs increase because of the frequent trading. Figure 9 shows that the no-trade ranges of DB become large, but those of other assets become small as the range of the bounds of the cash constraints (or the multiple α) becomes large. When the multiple α is more than one (which corresponds to the lower bound of 3% and the upper bound of 10%), the no-trade ranges become insensitive to the bounds of constraints.

3.3. The comparison of the optimal no-trade ranges with the actual weights

We examine the actual weights published quarterly from March 2008 to December 2012 by the GPIF¹⁸. Figure 10 shows the actual weights ('Actual W'), the policy asset mix ('PAM')

¹⁸The GPIF has managed their investment assets using the no-trade ranges in Table 1 since March 2008.

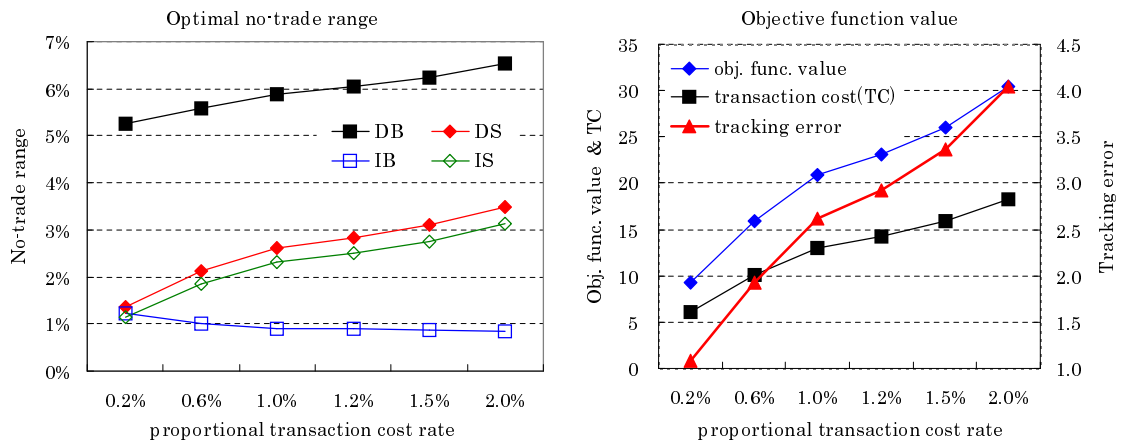


Figure 7: Sensitivity analysis(1) proportional transaction cost rate

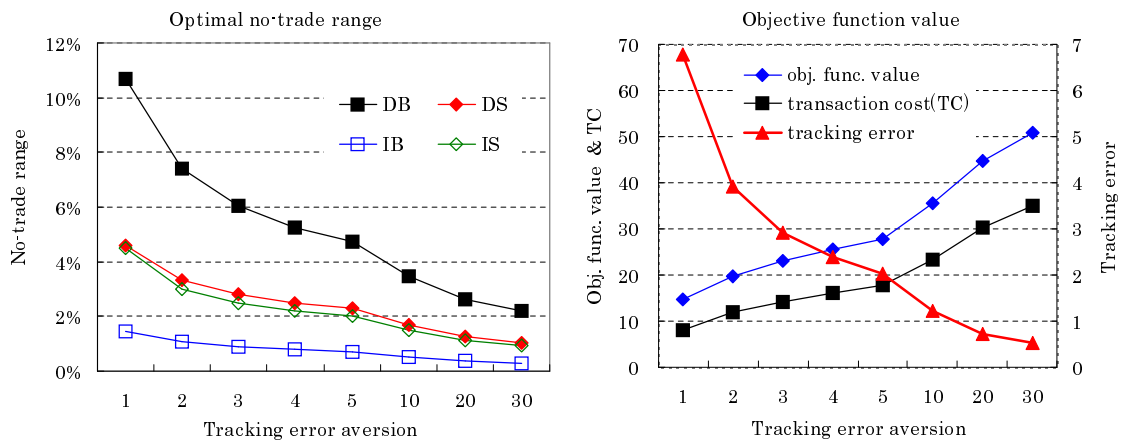


Figure 8: Sensitivity analysis(2) tracking error aversion

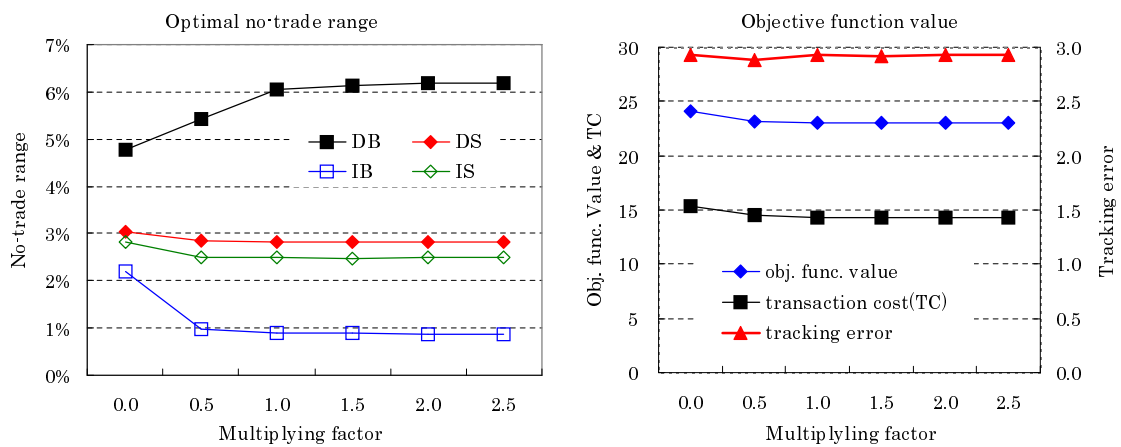


Figure 9: Sensitivity analysis(3) cash constraint parameter

and the no-trade ranges of the GPIF('GPIF'), and the optimal no-trade ranges for 5 and 50 year problems solved by the DFO method using the parameters of the basic analysis in Section 3.1, where the rebalancing interval is a month($\Delta t = \frac{1}{12}$).

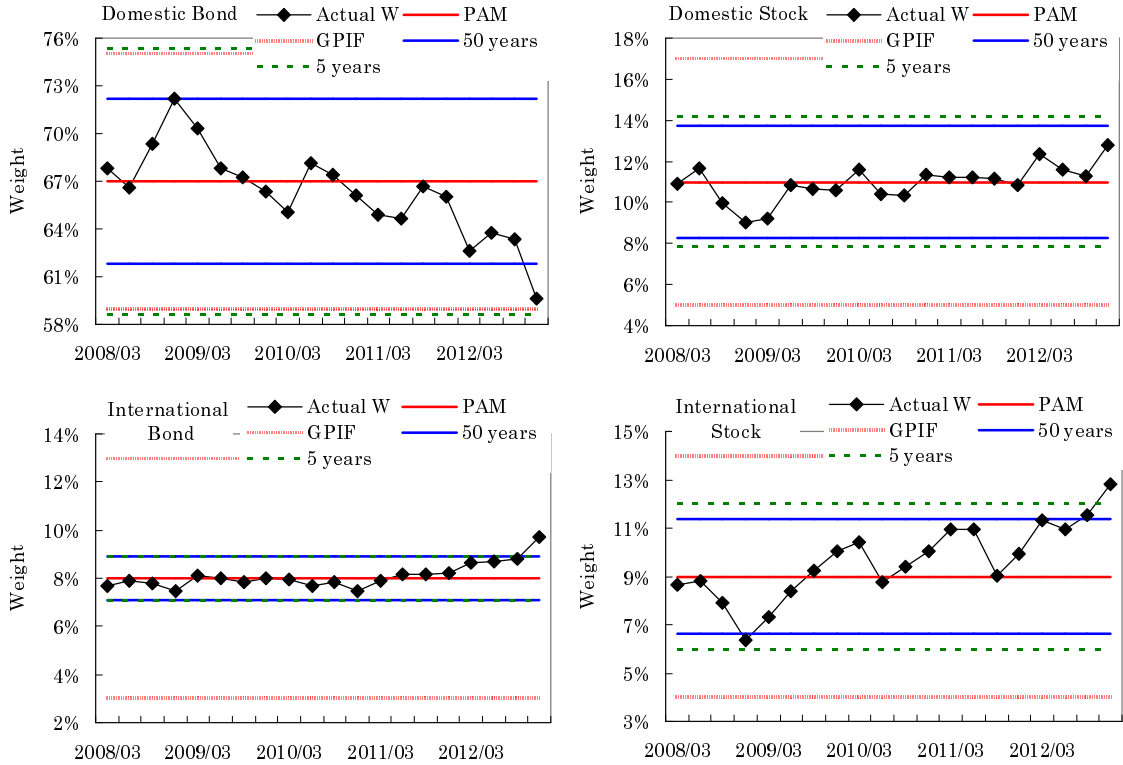


Figure 10: The optimal no-trade ranges and the actual weights

The actual weights have moved within the no-trade ranges set by the GPIF because their ranges are relatively large. The optimal no-trade ranges derived in this paper are smaller than those of the GPIF. The actual weights have moved almost just within the optimal no-trade ranges except four cases¹⁹, and thus the reasonable ranges can be derived to some extent²⁰.

4. Time-dependent no-trade ranges in a finite horizon

4.1. Finite horizon model

In this paper, we formulated the model in Section 2 and derived the optimal constant and symmetric no-trade ranges in the numerical analysis in Section 3 because it is easy to employ the rebalance rule in practice with the constant and symmetric no-trade ranges with respect to the policy asset mix. The results in Subsection 3.3 show the optimal no-trade ranges are reasonable in examining the actual weights. Meanwhile, the previous studies show that the optimal no-trade boundaries are time-dependent in a finite horizon. Then we derive

¹⁹Four cases are IS in December 2012 for 5 year problem, IS in September and December 2012 for 50 year problem, and DB and IB in December 2012 for 50 year problem. The probability of the violation is 1.25%(= 1/80) for 5 year problem, and 3.75%(= 3/80) for 50 year problem.

²⁰It is difficult to evaluate the validity of the optimal ranges because the optimal values depend on the parameters such as the proportional transaction costs and the tracking error aversions. This is one of the ways to evaluate the validity.

the optimal time-dependent and symmetric no-trade ranges, and compare them with the constant and symmetric ranges.

Hibiki *et al.*(2014) formulate the model with the time-dependent no-trade boundaries in the Leland approach(1999) which minimizes the objective function consisting of the transaction costs and the tracking errors. The boundaries are expressed by the exponential functions to show that the ranges between two boundaries become large as the time to maturity(time horizon) gets small. It is consistent with the previous studies for the problems with the CRRA utility function(Genotte and Jung(1994), Liu and Loewenstein(2002)). In this paper, we express the time-dependent no-trade boundaries $q_i(t)$ in Equation (4.1) by reference to Hibiki *et al.*(2014),

$$q_i(t) = \alpha_i \{1 + \gamma e^{-\beta(T-t+\Delta t)}\}, \quad (i = 1, \dots, N - 1), \quad (4.1)$$

where α_i , β and γ are the parameters determining the ranges. When $T \rightarrow \infty$,

$$\lim_{T \rightarrow \infty} q_i(t) = \alpha_i, \quad (4.2)$$

and the equations can express the constant boundaries in the infinite horizon. Moreover, they can express the various function forms by varying β and γ . When $t = \Delta t$, the boundaries are approximated by α_i as follows,

$$q_i(\Delta t) = \alpha_i (1 + \gamma e^{-\beta T}). \quad (4.3)$$

This shows the range of asset i at time Δt is almost the same as the value of α_i ²¹. Therefore, α_i is called the initial no-trade range. When $t = T$,

$$q_i(T) = \alpha_i (1 + \gamma e^{-\beta \Delta t}), \quad (4.4)$$

and the ranges at time T are affected by the value of γ .

The problem with $N + 1$ variables which consist of three kinds of variables α_i , β and γ is solved by the DFO method, and the optimal time-dependent and symmetric no-trade boundaries can be derived.

4.2. Numerical analysis in finite cases

We solve the problems for some combinations of four kinds of time intervals($\Delta t = 0.05, 0.1, 0.2, 0.5$) and four kinds of time horizons(5, 10, 15, 20 years), and derive the optimal time-dependent ranges by Equation (4.1). Figure 11 is shown to compare the results for the different time horizons. The horizontal axis is the time to maturity. The graphs above are the results for $\Delta t = 0.1$, and the graphs below are for $\Delta t = 0.5$. The left graphs are the results for the domestic assets, and the right graphs are for the international assets.

When the time intervals are the same, the no-trade range $q_i(t)$ is determined by the time to maturity regardless of the length of the time horizon except a domestic bond in 5 year problem. The no-trade ranges are almost the same until the time to maturity is about three years, but the ranges become large as the time to maturity gets small.

Figure 12 is shown to compare the results for the different time intervals. The graphs above are the results for 5 year problems, and the graphs below are for 20 year problems.

When the time interval is large, the no-trade range is reduced to avoid the increase in the tracking errors because the investors cannot trade assets except the time points in the discrete-time case.

²¹Technically, it depends on the values of β and γ .

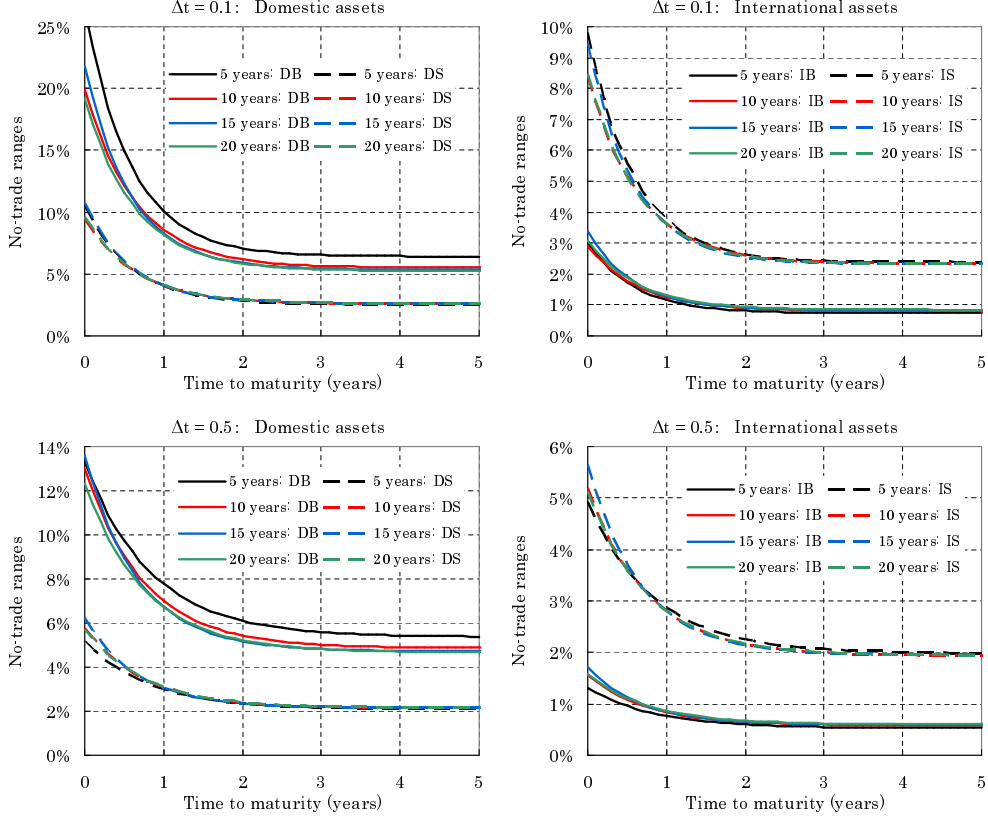


Figure 11: Results for the different time horizons in finite cases

We compare the time-dependent strategy with the constant strategy. Theoretically, the optimal ranges are time-dependent in the finite case, and constant in the infinite case. The no-trade ranges are almost the same when the time point is far from the maturity. It is expected that the initial no-trade ranges for the time-dependent strategy are close to the ranges for the constant strategy as the maturity is long because the ranges near the maturity are averaged over the whole periods. The plots of two values are shown to examine the relationship between them in Figure 13, where the horizontal axis is the constant no-trade range (q_i), and the vertical axis is the initial no-trade ranges for the time-dependent strategy (α_i). We solve the 50 year problem additionally to compare the constant strategy. Five plots for each time interval in the graph correspond to the time horizons of 5, 10, 15, 20 and 50 years. The points on the 45-degree line could be the same values.

The constant ranges are larger than the initial ranges for the time-dependent strategy. The shorter the maturity is, the larger the constant range is. The longer the maturity is, the closer both values are, and the plots become close to the 45-degree line. As the result, both optimal values are almost the same for 50 year problem. When we need to decide the ‘constant’ no-trade range by the finite horizon model for the time-dependent strategy, we can use the initial no-trade ranges for 50 year problem.

In Section 3, we solve the problem for the constant strategy to use the results in practice despite of the finite time horizon case. However, the objective function value increases more than the time-dependent strategy. We define the deterioration ratios of the objective function values in Equation (4.5), and show them in Figure 14.

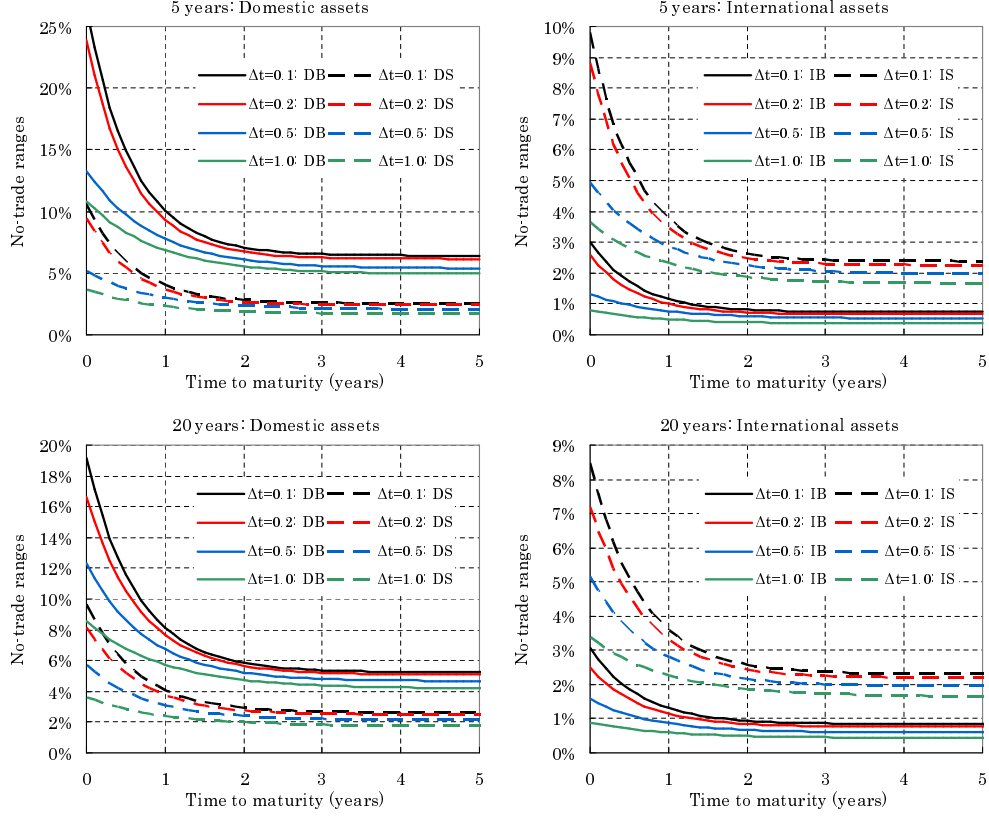


Figure 12: Results for the different time intervals in finite cases

$$\text{Deterioration ratio} = \frac{\text{Objective function value for constant strategy}}{\text{Objective function value for time-dependent strategy}} - 1 \quad (4.5)$$

The longer the maturity is, the smaller the deterioration ratio is. This reason is that the range for the constant strategy is closer to the initial range for the time-dependent strategy as the maturity becomes long. The ratio is about 2% for the case of $\Delta t = 0.1$ and 20 year horizon. Therefore, the constant strategy is much less disadvantaged than the time-dependent strategy as long as the maturity is more than 20 years.

5. Conclusion

In this paper, we derive the optimal solutions numerically for five asset problem with transaction costs and the symmetric no-trade ranges which were not solved in the previous literatures but are needed to solve in practice. In particular, we solve the optimal problem to decide the no-trade ranges according to the parameters used in the GPIF. It is difficult to evaluate the optimal no-trade ranges themselves compared to the ranges used in the GPIF directly because the GPIF does not publish how to decide the ranges. But, the actual weights have moved almost just within the optimal no-trade ranges derived by the DFO method, while the no-trade ranges set by the GPIF are relatively large in comparison with the actual weights. We can show the theoretical methodology and reasoning to decide the optimal no-trade ranges though they are different from the permissible ranges of the GPIF.

We may need to rebalance assets even within the no-trade ranges when the boundary constraints for cash are imposed. We clarify the fact and build the algorithm in the

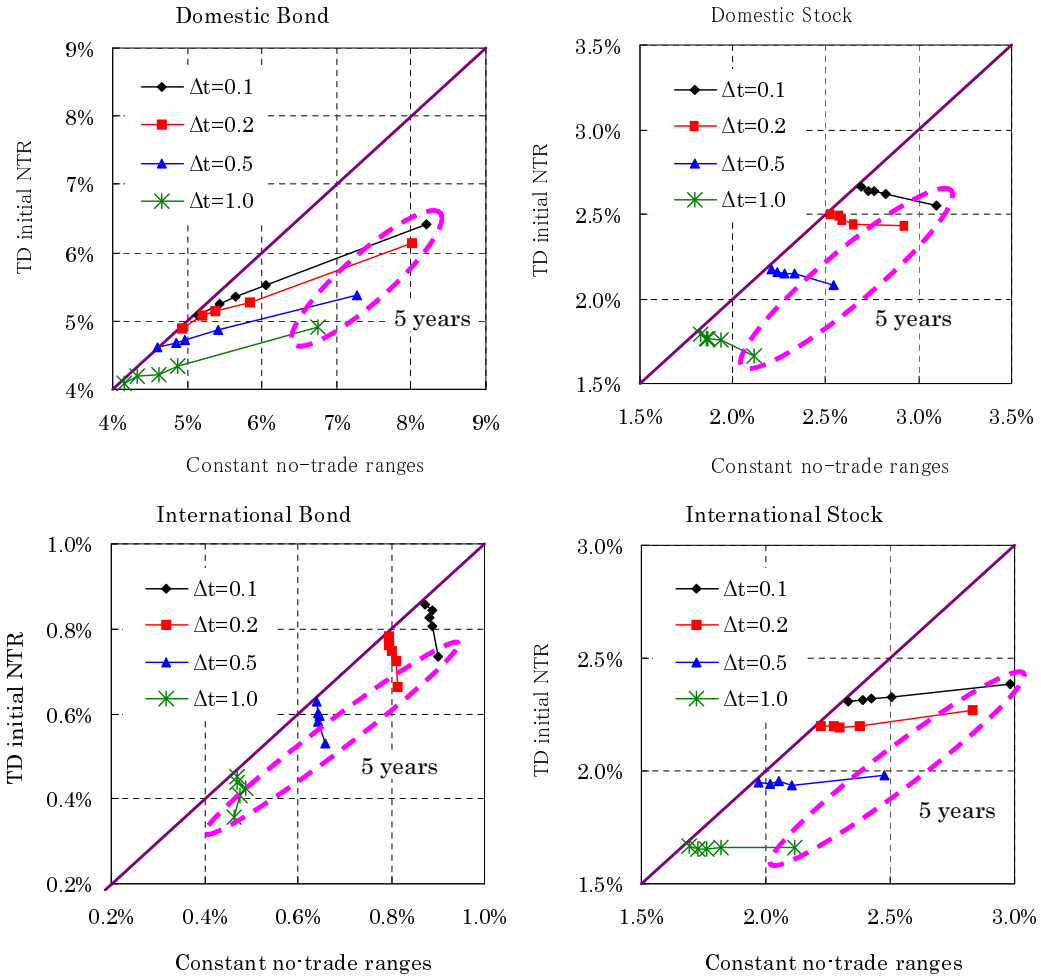


Figure 13: Comparison of the time-dependent strategy with the constant strategy

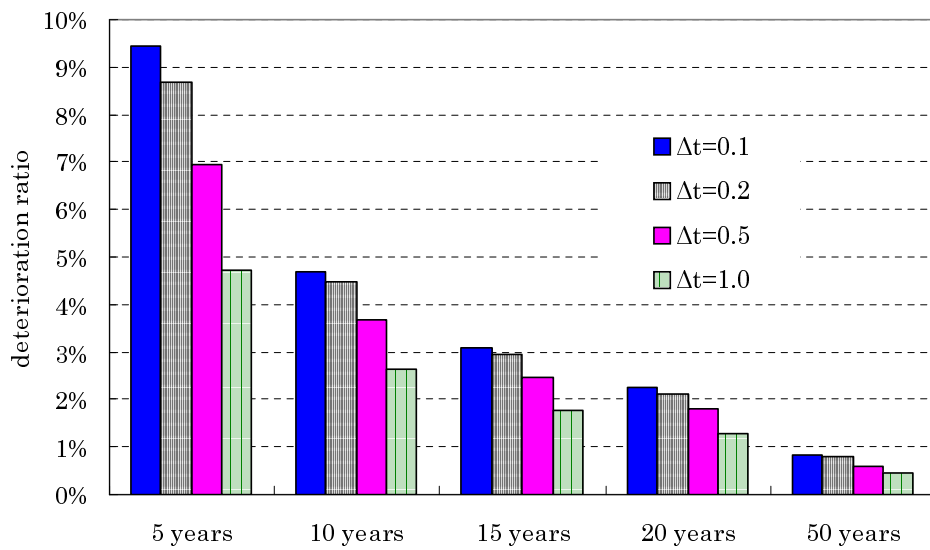


Figure 14: Deterioration ratio of objective function value

simulation procedure in the discrete-time model. We conduct the sensitivity analysis to examine the influence to the no-trade ranges for the different proportional transaction cost rates, tracking error aversions and cash constraints. In addition, we compare the optimal time-dependent no-trade ranges with the constant no-trade ranges. The numerical results show the possibilities of applying the DFO model to the practical problem determining the symmetric no-trade ranges.

In our paper, we assume we formulate the trade-off between transaction costs and tracking error under a given policy asset mix because the problem is defined, based on the Leland(1999) and Hibiki *et al.*(2014). However, we may need to solve the problem where the no-trade ranges and the target weights are simultaneously determined under the mean-variance objective. We would like to try to solve the problem in our next future research. On the other hand, we assume the asset returns are normally distributed in this paper because we develop the model based on Leland(1999) and the GPIF use the parameters for the mean-variance model. Various stochastic processes are used to represent the return process such as a time-series model, a jump model, and a model with copula among asset returns. In the future research, we need to adopt the more sophisticated model, and solve the optimal problem by the DFO approach.

Appendix

A. Comparison of the symmetric case with the general case

A parallelogram-like no-trade region is optimal for the asset allocation problem with transaction costs in general as in the left-hand side of Figure 15. On the other hand, we derive the optimal symmetric no-trade ranges (q_1, q_2) from a practical perspective, which is equivalent to rectangular regions where the policy asset mix (p_1, p_2) is the center of region as in the right-hand side of Figure 15.

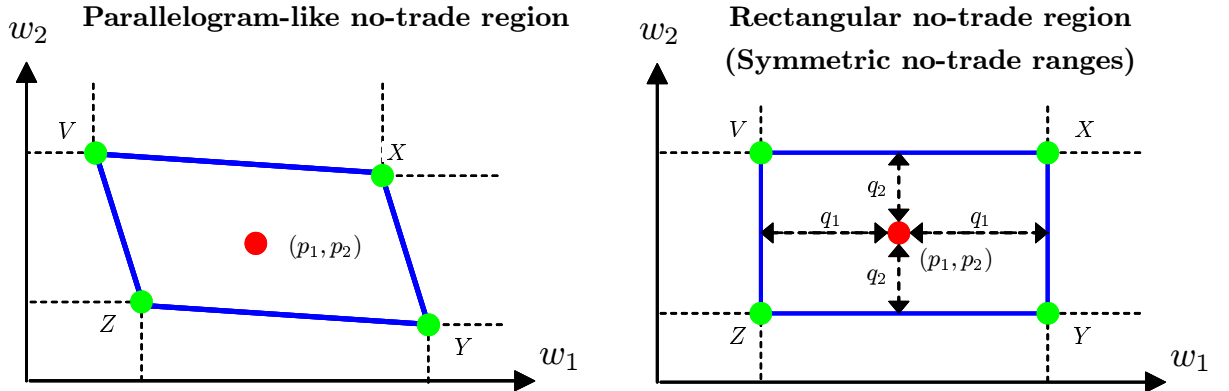


Figure 15: No-trade regions

In this section, we compare the constant and symmetric no-trade ranges in practical use with the general no-trade region for three-asset case(of two risky assets and a riskless asset). The baseline parameters in the analysis are shown in Table 4, used in Leland(1999) and Hibiki *et al.*(2014).

We conduct the sensitivity analysis of the proportional transaction costs, the correlation coefficients and the tracking error aversions. Ten kinds of values of each parameter in Table

Table 4: Baseline parameters

Expected rate of return	$\mu_1 = \mu_2 = 12.5\%$
Standard deviation	$\sigma_1 = \sigma_2 = 20\%$
Correlation	$\rho_{12} = 0.2$
Discount rate	$r = 7.5\%$
Proportional transaction costs	$k_1 = k_2 = 1\%$
Policy asset mix	$p_1^* = p_2^* = 0.4$
Tracking error aversion	$\lambda = 1.3$
Time interval and horizon	$\Delta t = 0.1, 10 \text{ years}$

5 are examined, and therefore thirty problems are solved in total²².

Table 5: Parameter values for the sensitivity analysis

	1	2	3	4	5	6	7	8	9	10
Transaction costs	0.1%	0.3%	0.5%	0.7%	<u>1%</u>	1.5%	2%	2.5%	3%	3.5%
Correlation	-0.3	-0.2	-0.1	0.0	0.1	<u>0.2</u>	0.3	0.4	0.5	0.6
TE aversion	0.5	1	<u>1.3</u>	2	3	4	5	10	20	30

※ Underlined parameters are baselines in Table 4.

A.1. General case(Hibiki *et al.*(2014))

The results solved by the DFO method for the three-asset case of Leland model(1999) are obtained in Hibiki *et al.*(2014). Figure 16 shows the no-trade regions where the origin is the policy asset mix for the sensitivity analysis, and Figure 17 shows the mean no-trade ranges from the corner points to policy asset mix, which may be compared with the symmetric no-trade ranges in Section A.2.

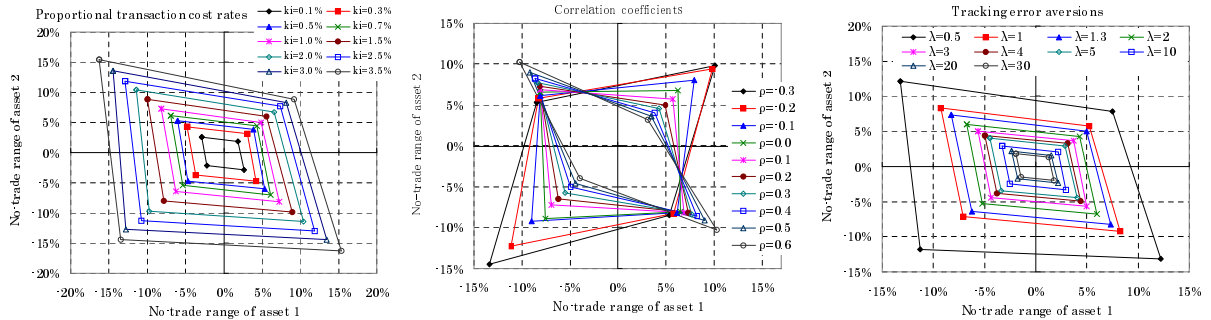


Figure 16: No-trade regions where the origin is the policy asset mix

As the proportional transaction cost rate becomes large, the no-trade region gets wider to avoid increasing the transaction cost. As the tracking error aversion becomes large, the no-trade region gets smaller to prevent losing touch with the policy asset mix. The shape of the no-trade region is a rectangle at zero correlation. The shape changes as the correlation deviates from zero. As the correlation becomes positively large, the shape of

²²When we examine the sensitivity of the proportional transaction costs, ten cases are solved under correlation of 0.2 and tracking error aversion of 1.3 as in Table 4.

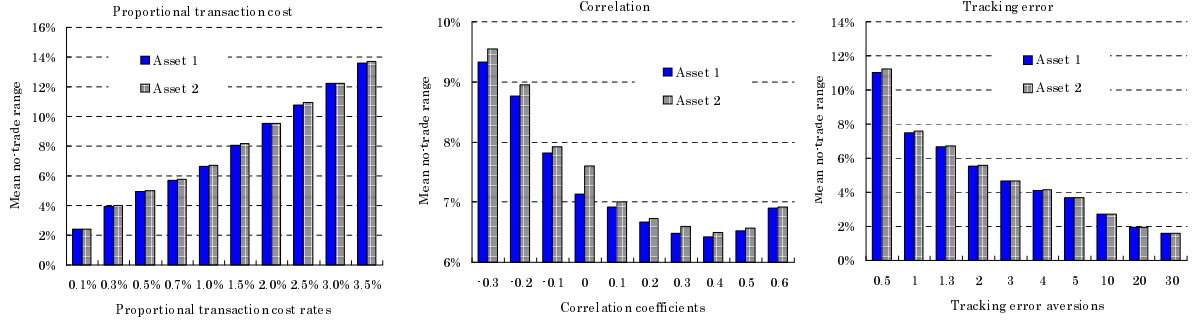


Figure 17: Mean no-trade ranges from the policy asset mix

no-trade region is like a parallelogram which expands from top left to bottom right. As the correlation becomes negatively large, the shape of no-trade region expands from top right to bottom left²³. When we change the parameters associated with the proportional transaction cost rates and the tracking error aversions, the mean no-trade ranges and the sizes of the no-trade regions vary in a similar way. However, the mean no-trade ranges are less sensitive to the correlation than other parameters while the shapes of the no-trade regions are more sensitive to the correlation.

A.2. Symmetric case

We calculate the optimal constant and symmetric no-trade ranges so that they can be obtained in the form of $p_i \pm q_i\%$, where p_i is the policy ratio of asset i and q_i is the no-trade range of asset i . As in Section A.1, we show the results of the sensitivity analysis derived by the DFO method in Figure 18.

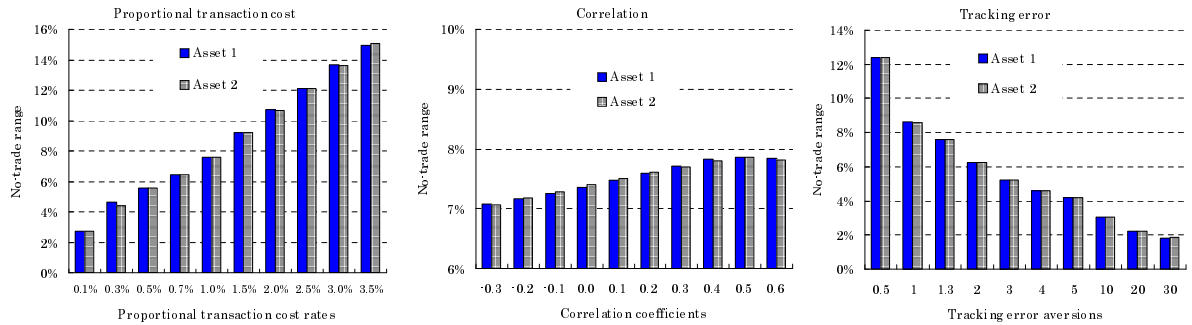


Figure 18: No-trade ranges from the policy asset mix

As the proportional transaction cost rate becomes large, the no-trade region gets larger. As the tracking error aversion becomes large, the no-trade region gets smaller to avoid losing touch with the policy portfolio. These results are consistent with those of the average of the general no-trade ranges. However, in the case of sensitivity analysis for the correlation, the results of the symmetric no-trade ranges are different from the mean ranges in Figure 17 because the shapes of the general no-trade regions are sensitive to the correlations. In particular, the symmetric no-trade ranges are larger as the correlations are larger in the cases of less than the correlation of 0.5. The symmetric no-trade ranges are smaller after

²³The shape of the no-trade region is square in the zero correlation case because the parameters of both assets are same. The shapes are like rhombus in the non-zero correlation cases.

peaking at the correlation of 0.5. However, the symmetric no-trade ranges are not sensitive to the correlations. Therefore, we cannot reflect the influence of the different correlation to the no-trade ranges in the case that the no-trade ranges are symmetric.

The objective function values for the symmetric cases are larger than those for the general cases because the symmetric problems are solved under the constraints of the symmetric no-trade ranges. We show the increasing rates of the objective function values in Figure 19.

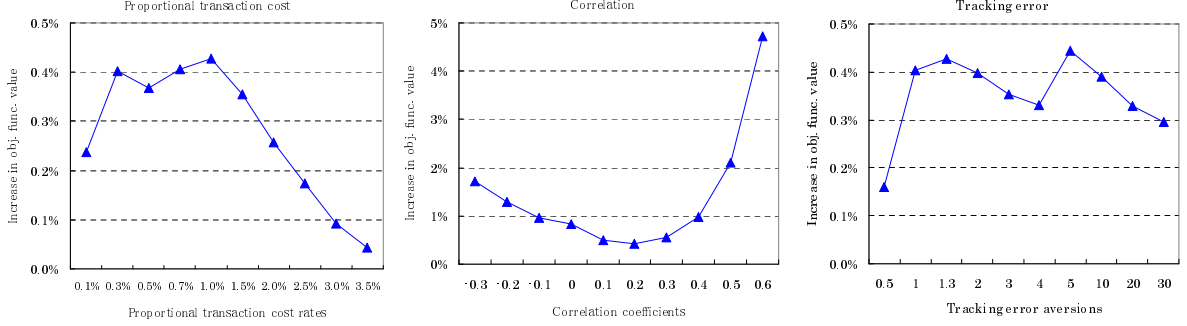


Figure 19: Increasing rates of the objective function values

The objective function values increase only about 0.5% in the case of the sensitivity analysis for the proportional transaction costs and the tracking error aversions. However, the objective function values increase between 0.5% and 5% in the case of the sensitivity analysis for the correlations. As the absolute values of correlations are large, the objective function values are larger because the no-trade regions change shape from the rectangle to the parallelogram. But the maximum increase in the objective function value is about 2% between -0.3 and 0.5 correlation coefficients.

Finally, we discuss the computation time. Average computation time is about 17 minutes for 30 general cases. Though it is acceptable computation time to solve the decision problem, we can solve the problem to decide the symmetric no-trade ranges only in about 90 seconds on average. Computation time decreases about 90%. The fact shows the use of the symmetric no-trade ranges is effective in the computational aspect.

B. Tracking error and mean-variance utility

We explain the rationale for modeling the trade-off between tracking error and transaction costs in our model by showing the relationship between tracking error and mean-variance utility in accordance with Leland(1999). At first, the mean-variance utility for N -asset case can be expressed as

$$U(\mathbf{w}) = \boldsymbol{\mu}^\top \mathbf{w} - \lambda \mathbf{w}^\top \mathbf{V} \mathbf{w}, \quad (\text{B.1})$$

where $\boldsymbol{\mu} \in \mathbb{R}^N$ is an expected return vector, $\mathbf{V} \in \mathbb{R}^{N \times N}$ is a covariance matrix, $\mathbf{w} \in \mathbb{R}^N$ is a weight vector, and λ is a risk aversion.

The mean-variance utility maximization problem can be formulated as

$$\text{Maximize } U(\mathbf{w}), \quad (\text{B.2})$$

$$\text{subject to } \mathbf{e}^\top \mathbf{w} = 1, \quad (\text{B.3})$$

where $\mathbf{e} = (1, 1, \dots, 1)^\top \in \mathbb{R}^N$. The optimal solutions \mathbf{w}^* can be found using Lagrangian multiplier method as follow,

$$\mathbf{w}^* = \frac{1}{2\lambda} \mathbf{V}^{-1} (\boldsymbol{\mu} - \eta \mathbf{e}), \quad \text{where } \eta = \frac{\mathbf{e}^\top \mathbf{V}^{-1} \boldsymbol{\mu} - 2\lambda}{\mathbf{e}^\top \mathbf{V}^{-1} \mathbf{e}}. \quad (\text{B.4})$$

Suppose that the weight vector at time τ is $\mathbf{w}(\tau) \in \mathbb{R}^N$, and the trading weight vector is $|\Delta\mathbf{w}|$ ²⁴. The mean-variance utility can be expressed as

$$U(\mathbf{w}(\tau)) = (\boldsymbol{\mu}^\top \mathbf{w}(\tau) - \mathbf{k}^\top |\Delta\mathbf{w}|) - \lambda \mathbf{w}(\tau)^\top \mathbf{V} \mathbf{w}(\tau), \quad (\text{B.5})$$

for the problem with the proportional transaction cost because the transaction cost $\mathbf{k}^\top |\Delta\mathbf{w}|$ is subtracted from the expected portfolio return $\boldsymbol{\mu}^\top \mathbf{w}(\tau)$.

In this situation, investors will minimize the utility loss which is the difference between the mean-variance utility at \mathbf{w}^* and at $\mathbf{w}(\tau)$, and it can be expressed in what follows.

$$\begin{aligned} UL &= U(\mathbf{w}^*) - U(\mathbf{w}(\tau)) \\ &= (\boldsymbol{\mu}^\top \mathbf{w}^* - \lambda \mathbf{w}^{*\top} \mathbf{V} \mathbf{w}^*) - (\boldsymbol{\mu}^\top \mathbf{w}(\tau) - \mathbf{k}^\top |\Delta\mathbf{w}| - \lambda \mathbf{w}(\tau)^\top \mathbf{V} \mathbf{w}(\tau)) \\ &= \boldsymbol{\mu}^\top (\mathbf{w}^* - \mathbf{w}(\tau)) - \lambda \mathbf{w}^{*\top} \mathbf{V} \mathbf{w}^* + \lambda \mathbf{w}(\tau)^\top \mathbf{V} \mathbf{w}(\tau) + \mathbf{k}^\top |\Delta\mathbf{w}| \end{aligned} \quad (\text{B.6})$$

We substitute the optimal solutions (B.4) into Equation (B.6), and we obtain the following equation.

$$UL = \lambda (\mathbf{w}(\tau) - \mathbf{w}^*)^\top \mathbf{V} (\mathbf{w}(\tau) - \mathbf{w}^*) + \mathbf{k}^\top |\Delta\mathbf{w}| \quad (\text{B.7})$$

The first term of Equation (B.7) shows tracking error, and the second term shows transaction cost. Though we need to pay attention to the fact that this can be applied to the simple mean-variance model, this is the rationale for modeling the trade-off between tracking error and transaction cost in our model as well as the Leland model(1999). Moreover, this would imply that we may choose the same “ λ ” as the risk aversion for the tracking error aversion.

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²⁴The weight vector before trading assets is either $\mathbf{w}(\tau) + \Delta\mathbf{w}$ or $\mathbf{w}(\tau) - \Delta\mathbf{w}$ at time τ .

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